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FREDHOLM THEORY OF PARTIAL DIFFERENTIAL EQUATIONS ON COMPLETE RIEMANNIAN MANIFOLDS

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This paper studies necessary and sufficient conditions for differential operators to be Fredholm on the Sobolev spaces of a complete (not necessarily compact) Riemannian manifold Ω. The conditions are formulated algebraically in terms of the nonvanishing of the operator's principal symbol on Ω (ellipticity) and its "total symbol" at infinity of Ω. The operators considered arise by taking sums of products of vector fields, all of whose covariant derivatives vanish at infinity; and the study involves C^*-algebra techniques. The required technical restrictions on the curvature and topology of Ω near infinity are much weaker than those in earlier joint work with H. O. Cordes.

0. Introduction. Let Ω be an n-dimensional paracompact C^∞-manifold with complete Riemannian metric $ds^2 = g_{ij}dx^i dx^j$ and surface measure $d\mu = \sqrt{g}\,dx$ where $g = \det(g_{ij})$. Let $\Delta = g^{ij}\nabla_i\nabla_j$, the Beltrami-Laplace operator on Ω, where $(g^{ij}) = (g_{ij})^{-1}$ and ∇ denotes covariant differentiation with respect to the Riemannian connection. Then $\Lambda = (1-\Delta)^{-1/2}$ is a positive-definite operator in $\mathscr{L}(\mathfrak{H})$, the bounded operators over the Hilbert space $\mathfrak{H} = L^2(\Omega, d\mu)$. Define the Nth-Sobolev space $\mathfrak{H}_N \subset \mathfrak{H}$ for $N = 1, 2, \cdots$ by requiring $\Lambda^N\colon \mathfrak{H} \to \mathfrak{H}_N$ to be an isometric isomorphism. It was shown in [3] that $C_0^\infty(\Omega)$ is dense in each $\mathfrak{H}_N$.

Now suppose we are given a differential operator L on Ω, of order N, such that we obtain a bounded map $L\colon \mathfrak{H}_N \to \mathfrak{H}$. We may ask the question when is L Fredholm (i.e., when are ker L and coker L finite-dimensional subspaces of $\mathfrak{H}_N$ and $\mathfrak{H}$ respectively)? If Ω is compact, Seeley [13] showed that L is Fredholm if and only if L is elliptic (i.e., the principal symbol of L never vanishes on the cosphere bundle $S^*\Omega$). For the case $\Omega = \boldsymbol{R}^n$, on the other hand, ellipticity is not sufficient to imply L is Fredholm; Cordes and Herman [4] derived necessary and sufficient Fredholm criteria in terms of the "total symbol" of L.

In [4] the techniques involved considering operators $L\Lambda^N$ as generators of a C^*-algebra $\mathfrak{A} \subset \mathscr{L}(\mathfrak{H})$ which is commutative modulo the compact ideal, $\mathscr{K}$. The symbol of $L\Lambda^N$ is then defined as the continuous function $\sigma_{L\Lambda^N}$ on the maximal ideal space $\boldsymbol{M}$ of $\mathfrak{A}/\mathscr{K}$ provided by the Gel'fand theory. Thus $L\Lambda^N$ is Fredholm if and only