PACIFIC JOURNAL OF MATHEMATICS
Vol. 87, No. 1, 1980

NEIGHBORLY BUSHES AND THE RADON-NIKODÝM PROPERTY FOR BANACH SPACES

PHILIP W. MCCARTNEY

We introduce the neighborly bush property (NBP). A dual Banach space has the NBP if and only if it contains a bush, but there is a Banach space with the NBP that does not contain a bush and therefore has the Radon-Nikodým property. Any Banach space with the NBP has a nonseparable second dual. Other related results are obtained.

A Banach space X has the Radon-Nikodým property (RNP) if and only if, for every finite measure space (Ω, Σ, μ), and each μ-continuous, X-valued measure m on Σ of bounded variation, there exists a Bochner-integrable function $f: \Omega \to X$ such that

$$m(E) = \int_E f d\mu \quad \text{if } E \text{ is in } \Sigma \text{ .}$$

For convenience in defining "tree," we introduce the set $S = \{(k, j): k \in N, j \in N, 1 \leqq j \leqq 2^{k-1}\}$, where N denotes the set of positive integers. Also, for each positive integer n, let $S_n = \{(k, j) \in S: k \leqq n\}$.

DEFINITION 1. Let X be a Banach space and ε a positive number. A *tree with separation constant* ε is a function $T: S \to X$ such that, for every (k, j) in S,

(i) $T(k, j) = 1/2[T(k+1, 2j) + T(k+1, 2j-1)]$ (*Averaging Property*),

(ii) $\|T(k, j)\| \leqq 1$,

(iii) $\|T(k+1, 2j) - T(k+1, 2j-1)\| > \varepsilon$ (*Separation Property*).

We denote $T(k, j)$ by $x^{k,j}$, and call $x^{k+1,2j-1}$ and $x^{k+1,2j}$ the *branch points* of T corresponding to the *node* $x^{k,j}$. The connection between the RNP and trees seems to have been noted first (implicitly) in [9] and explicitly in [11, page 222]. It follows easily from dentability that if a Banach space contains a tree, it does not have the RNP [3, page 127]. A dual space X has the RNP if and only if X does not contain a tree ([3, page 127], [11, page 222]). A Banach space has the RNP if and only if it does not contain a bush [3, page 216]. The reader who is interested in further historical information should see [1], [2], [3], and [5]. There are many other characterizations of the RNP discussed in [1], [2], and [3].