

THE FULL C^* -ALGEBRA OF THE FREE GROUP ON TWO GENERATORS

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$C^*(F_2)$ is a primitive C^* -algebra with no nontrivial projection. $C^*(F_2)$ has a separating family of finite-dimensional representations.

1. Introduction. We present a "generic" C^* -algebra in illustration of several peculiar phenomena that may occur in the theory of representations.

Let F_2 denote the free group on two generators. If π is the universal unitary representation of F_2 on a Hilbert space \mathcal{H} , then the full group C^* -algebra $C^*(F_2)$ is the C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the set $\{\pi(g): g \in F_2\}$ (see [4, § 13.9]). Alternatively, we can re-define $C^*(F_2)$, in an operator-theoretical setting, as follows:

DEFINITION. Let U, V be two unitary operators on a Hilbert space \mathcal{H} . We say that (U, V) is a *universal pair of unitaries* iff for each pair of unitary operators (U_1, V_1) on a Hilbert space \mathcal{H}_1 , the assignment

$$\begin{cases} U \longmapsto U_1 \\ V \longmapsto V_1 \end{cases}$$

extends to a $*$ -homomorphism from $C^*(U, V)$ onto $C^*(U_1, V_1)$.

DEFINITION. We let $C^*(F_2)$ denote the abstract C^* -algebra which is $*$ -isomorphic with the C^* -algebra generated by a universal pair of unitaries.

Obviously, the universal pairs of unitaries are *unique* up to algebraic $*$ -isomorphic equivalence. To see the existence of a universal pair of unitaries, we may simply let

$$(*) \quad U = \bigoplus U_\nu, \quad V = \bigoplus V_\nu$$

where (U_ν, V_ν) runs through all possible pairs of unitary operators on a fixed separable Hilbert space. By some judicious selection, it suffices to let ν run through only a countable index set. [In fact, for a general separable C^* -algebra $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$. There is always a projection P of countable dimension, such that $A \mapsto PAP$ is a $*$ -isomorphism from \mathfrak{A} onto $P\mathfrak{A}P \subseteq \mathcal{B}(P\mathcal{H})$.]

The main result of this paper is concerned with various expres-