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COBORDISM OF BRANCHED COVERING SPACES

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We shall develop a general method of constructing branched covering spaces of spheres and other manifolds. In the case of the sphere, this method gives rise via transversality to an equivalence between certain cobordism classes of branched covering spaces and the homotopy groups of certain topological spaces. We will interpret the cobordism equivalence and compute the homotopy groups in one case.

Let (X^n, B^{n-2}) be a pair of finite CW complexes such that X^n is a PL n-manifold in a neighborhood of B^{n-2} and B^{n-2} is embedded as a locally flat submanifold. Suppose $\varphi: \pi_1(X^n - B^{n-2}) \rightarrow \sum(q)$ is a representation into the permutation group of q letters. (Because we shall identify two such representations if they differ by an inner automorphism of $\Sigma(q)$ there is no need to choose a base point.) Now let Y^k be a closed PL manifold and let $f: Y^k \rightarrow X^n$ be a PL map in general position with respect to B^{n-2}. The map $\varphi f_*: \pi_1(Y^k - f^{-1}(B^{n-2})) \rightarrow \Sigma(q)$ is a representation of $\pi_1(Y^k - f^{-1}(B^{n-2}))$ into $\Sigma(q)$ defined up to an inner automorphism of $\Sigma(q)$. Such an equivalence class of representations defines a q-fold branched covering space $p: M^k(f) \rightarrow Y^k$, branched over the codimension two submanifold $f^{-1}(B^{n-2})$ (see [5]). A homotopy F between two such maps $f, g: Y^k \rightarrow X^n$, that is itself in general position with respect to B^{n-2}, gives rise to a cobordism $p: W^{k+1}(F) \rightarrow Y^k \times I$, branched over the codimension two submanifold $F^{-1}(B^{n-2})$. (We allow branched coverings $p: M^k(f) \rightarrow Y^k$ where $M^k(f)$ is disconnected and we allow the branch set to be empty. Thus, if f is the constant map to a point not on B^{n-2} then $M^k(f)$ is q disjoint copies of Y^k each mapped to Y^k by the identity.)

Two natural questions to ask at this point are:

1. What q-fold branched covering spaces $p: M^k(f) \rightarrow Y^k$ can be obtained in this manner?
2. What cobordisms between q-fold branched covering spaces can be obtained?

For at least one set of fixed X^n, B^{n-2}, Y^k, q and φ, these questions have geometrically interesting answers. In this case we interpret the cobordism, compute the homotopy group and present the results in the last section.

2. Construction of $X(n)$.

Let $A(n)$ denote the subgroup of even permutations of $\Sigma(n)$, the group of permutations of the