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$[0, \infty]$-VALUED, TRANSLATION INVARIANT MEASURES ON N AND THE DEDEKIND COMPLETION OF $^*R$

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This paper investigates $\{0, \infty\}$-valued, translation invariant measures on the set N of positive integers. The main tool in this investigation is Nonstandard Analysis and especially the completion, $^\sharp R$, in the sense of Dedekind of the Nonstandard Reals, $^*R$. The algebraic and topological properties of $^\sharp R$ are developed and exploited to obtain a classification theorem for a particularly nice class of $\{0, \infty\}$-valued, translation invariant measures on N.

1. Introduction. One of the basic problems in mathematics is to define a measure for a suitable collection of subsets of a given set X. When X is the real line there is a unique (up to scale) natural, countably additive, translation invariant measure, namely Lebesgue measure. However, if X is the set of positive integers, $N = \{1, 2, 3, \cdots\}$ then the situation is not so neat. First, since N is countable the only (up to scale) countably additive, translation invariant measure is the measure which assigns $+\infty$ to every infinite set and to every finite set assigns its cardinality. Although, this measure is very important it fails to distinguish between infinite sets in any way. One way to obtain a measure which does make some distinction between different infinite sets is to apply Zorn's lemma to find a nonprincipal ultrafilter $\mathscr{D}$ on N. Such an ultrafilter is a collection of subsets of N satisfying the following properties

(1) $\varnothing \notin \mathscr{D}$, $N \in \mathscr{D}$

(2) $A \in \mathscr{D}$, $A \subseteqq B \subseteqq N$ implies $B \in \mathscr{D}$

(3) $A, B \in \mathscr{D}$ implies $A \cap B \in \mathscr{D}$

(4) $\cap\{A \mid A \in \mathscr{D}\} = \varnothing$

(5) $\mathscr{D}$ is maximal with respect to properties (1)-(3).

Properties (1)-(3) are the defining properties for a filter on N. Property (4) says that the filter is not principal, and Property (5) says that $\mathscr{D}$ is an ultrafilter. Property (5) is equivalent to (5′).

$$(5') \; A \cup B \in \mathscr{D} \quad \text{implies} \quad A \in \mathscr{D} \quad \text{or} \quad B \in \mathscr{D} .$$

Intuitively, one thinks of the sets in $\mathscr{D}$ as "big" and assigns them measure one while the sets outside of $\mathscr{D}$ are given measure zero. This yields a finitely additive $\{0, 1\}$-valued measure which is extremely useful for many purposes. However, this measure lacks