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LINEAR OPERATORS FOR WHICH T^*T AND TT^* COMMUTE III

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Let T be a bounded linear operator on a Hilbert space H. Let $[T] = T^*T - TT^*$. The structure of operators such that T^*T and TT^* commute and rank$[T] < \infty$ is studied.

1. Introduction.

Let T be a bounded linear operator acting on a separable Hilbert space H. Let $[T] = T^*T - TT^*$ and $(BN) = \{T \mid T^*T$ and TT^* commute$\}$. As in [1] let $(BN)^+ = \{T \mid T \in (BN)$ and T is hyponormal$\}$.

In [2] it is shown that if $T \in (BN)$ and rank $[T] = 1$, (hence either T or T^* is in (BN^+), then $T = T_1 \oplus T_2$ where T_1 is normal and T_2 is a special type of weighted bilateral shift.

The purpose of this note is to examine the extension of this result to those $T \in (BN)$ for which rank $[T] < \infty$. The simple example [1]

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}, \quad TT^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad T^*T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

of a $T \in (BN)$, $T^2 \notin (BN)$, shows that if rank $[T] = 2$, then different behavior is possible.

2. Notation and preliminary results.

The notation of this section will be kept throughout the paper. Suppose that $T \in (BN)$ and rank $[T] = r$. Then, for the correct choice of orthonormal basis we have

$$T^*T = \begin{bmatrix} D_1 & 0 \\ 0 & P \end{bmatrix}, \quad TT^* = \begin{bmatrix} D_2 & 0 \\ 0 & P \end{bmatrix} \tag{1}$$

where $D_1 = \mathrm{Diag}\{\alpha_1, \cdots, \alpha_r\}$, $D_2 = \mathrm{Diag}\{\beta_1, \cdots, \beta_r\}$ with $\alpha_i \neq \beta_i$ for all i. Let $T = U(T^*T)^{1/2}$ be the polar factorization of T. Thus U is a partial isometry with $R(U) = R(T)$, $N(U) = N(T)$. Note that $U(T^*T)^{1/2} = (TT^*)^{1/2}U = T$ and T^*T and TT^* have identical spectrum except for zero eigenvalues. Also UU^* is the orthogonal projection onto $R(T) = N(T^*)^\perp$ while U^*U is the orthogonal projection onto $R(T^*) = N(T)^\perp$. Now $(T^*T)^{1/2} = U^*(TT^*)^{1/2}U$. Thus for any polynomial $p(\lambda)$,

$$p((T^*T)^{1/2}) = U^*p((TT^*)^{1/2})U + p(0)(I - U^*U)\,, \tag{2}$$