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THE TORSION GROUP OF A RADICAL EXTENSION

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The torsion group of a radical field extension is defined and its structure determined using a theorem of Kneser. In the case of a number field, a representation theorem is proved characterizing all abelian groups that can appear as torsion groups of a radical extension.

Let F be a field with multiplicative group F^*. Let K be an extension of F and let $T(K^*/F^*)$ be the torsion subgroup of K^*/F^*. In this paper we will determine the structure of the group $T(K^*/F^*)$ in case $K = F(\alpha)$ where α is a root of irreducible $x^m - a \in F[x]$ and char $F \nmid m$. In particular, we shall prove the following:

THEOREM A. *For a positive integer q, let ζ_q denote a primitive qth root of unity. Let $x^m - a \in F[x]$ be irreducible with root α and $m = 2^n m_0$ with m_0 odd. For p a prime let T_p denote the p-torsion of $T = T(F(\alpha)^*/F^*)$. Let $\eta_{2^n} = \zeta_{2^n} + \zeta_{2^n}^{-1}$. Define N to be the largest integer, if such exists, so that $\eta_{2^N} \in F$; otherwise, let $N = \infty$. Then*

$$T = \langle \alpha^{2^n} F^* \rangle \times T_2 \times H$$

where

(a) $H = \langle \{\zeta_k \in F(\alpha)\colon \textit{if } p \textit{ is prime and } p \mid k, \textit{ then } \zeta_p \notin F\} F^* \rangle$;

(b) *If* $\zeta_4 \notin F(\alpha) \backslash F$, *then* $T_2 = \langle \alpha^{m_0} F^* \rangle$;

(c) *If* $\zeta_4 \in F(\alpha) \backslash F$, *then*

(i) *if* $N = \infty$, $T_2 = \langle \alpha^{m_0} \zeta_{2^{n+1}} F^* \rangle \times \langle \{\zeta_{2^t}\colon \textit{all } t\} F^* \rangle \cong \boldsymbol{Z}_{2^{n-1}} \times \boldsymbol{Z}_{2^\infty}$;

(ii) *if* $n \leqq N < \infty$, $T_2 = \langle \alpha^{m_0} (1 + \zeta_{2^N})^{2^{N-n}} F^* \rangle \times \langle (1 + \zeta_{2^N}) F^* \rangle \cong \boldsymbol{Z}_{2^{n-1}} \times \boldsymbol{Z}_{2^N}$;

(iii) *if* $N < n$, $T_2 = \langle \alpha^{m_0} F^* \rangle \times \langle \alpha^{m_0 2^{n-N}} (1 + \zeta_{2^N}) F^* \rangle \cong \boldsymbol{Z}_{2^n} \times \boldsymbol{Z}_{2^{N-1}}$.

The following questions concerning the group $T(K^*/F^*)$ have already been examined for various extensions K/F:

(1) Let M be a subgroup of T containing F^* such that M/F^* is finite. When is it the case that $[M\colon F^*] = [F(M)\colon F]$? Kummer theory [6, p. 218] says that this equation holds when the exponent of M/F^* is m and a primitive mth root of unity ζ_m is an element of F. Besicovitch [1], Mordell [9], and Siegel [13] found necessary and sufficient conditions for this equation to hold in case the only roots of unity in $F(M)$ are ± 1. Kneser [5] has generalized all of these results by proving the

THEOREM. *The equation $[M\colon F^*] = [F(M)\colon F]$ holds iff, for every*