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POLYNOMIAL NEAR-FIELDS?

YONG-SIAN SO

It is well known that all finite fields can be obtained as homomorphic images of polynomial rings. Hence it is natural to raise the question, which near-fields arise as homomorphic images of polynomial near-rings.

It is the purpose of this paper to give the surprising answer: one gets no proper near-fields at all—in dramatic contrast to ring and field theory. Another surprising result is the fact that all near-fields contained in the near-rings of polynomials are actually fields.

Homomorphic images are essentially factor structures. So we take a commutative ring R with identity, from the near-ring $R[x]$ of all polynomials over R (or the near-ring $R_0[x]$ of all polynomials without constant term over R) and look for ideals I such that $R[x]/I$ becomes a near field. With this notation (and containing the one of [1] and [2]) we get our main result:

THEOREM 1. *If $R[x]/I$ (or $R_0[x]/I$) is a near-field then it is isomorphic to R/M (where M is a maximal ideal of R) and hence a field.*

The proof requires a series of lemmas as well as a number of results on near-fields.

Our first reduction is the one of $R[x]$ to $R_0[x]$.

LEMMA 1. *If I is an ideal of (the near-ring) $R[x]$ such that $R[x]/I$ is a near-field, then there exists an ideal J of $R_0[x]$ with $R[x]/I \cong R_0[x]/J$.*

Proof. $R_0[x] \subseteqq I$ implies $x \in I$, hence $R[x] \subseteqq I$, a contradiction. So we have $R_0[x] \nsubseteqq I$ and—since I must be maximal in order to get a near-field—$R_0[x] + I = R[x]$. By a version of the isomorphic theorem (which is valid in our case) we get

$$R[x]/I = (R_0[x] + I)/I \cong R_0[x]/(I \cap R_0[x])$$

and $J := R_0[x] \cap I$ will do the job.

REMARK 1. The converse of Lemma 1 does not hold: Take $J := \{a_2x^2 + a_3x^3 + \cdots + a_nx^n / n \in \boldsymbol{N},\ n \geqq 2,\ a_i \in \boldsymbol{R}\}$. Then $\boldsymbol{R}_0[x]/J \cong \boldsymbol{R}$ is a (near) field, but the near-ring $\boldsymbol{R}[x]$ is simple ([2] or [3], 7.89), so there is no $I \trianglelefteq \boldsymbol{R}[x]$ with $\boldsymbol{R}[x]/I \cong \boldsymbol{R}$.