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RIGHT CHAIN RINGS AND THE GENERALIZED SEMIGROUP OF DIVISIBILITY

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Let R be a ring with unit element and without zero-divisors and let $\tilde{H}(R) = \{\tilde{x} \mid 0 \neq x \in R\}$ where $\tilde{x}$ is the mapping from the set of all nonzero principal right ideals of R into itself defined by $\tilde{x}(aR) = xaR$. $\tilde{H}(R)$ is a partially ordered semigroup that can be considered as a generalization of the group of divisibility of a commutative integral domain. We study those rings R for which $\tilde{H}(R)$ is totally ordered.

1. Introduction. Associated with any commutative integral domain A is the partially ordered group $G(A)$ of nonzero fractional principal ideals of A with $aA \leqq bA$ if and only if aA contains bA. It is well known (see [4], [5], [8]) that $G(A)$, the group of divisibility, reflects certain properties of A, like A being a unique factorization domain, the fact that any two elements in A have a greatest common divisor or A being a valuation ring. This concept of a group of divisibility cannot be extended directly to a not necessarily commutative integral domain R.

In this paper we associate with any ring R with unit element and without zero-divisors a partially ordered semigroup $\tilde{H}(R)$ which is isomorphic to the semigroup $H(A) \subseteqq G(A)$ of nonzero principal ideals aA in A if A is a commutative domain.

After observing some basic facts about $\tilde{H}(R)$ we characterize in §3 those rings R with $\tilde{H}(R)$ totally ordered as right chain rings R with $Ja \subseteqq aR$ for all a in R and $J = J(R)$ the Jacobson radical of R. These rings are localizations of right invariant right chain rings. The main result of §4 is the theorem that a ring with $\tilde{H}(R)$ totally ordered and d.c.c. for prime ideals is right invariant. In a final §5 we show by examples that for every totally ordered group G there exists a ring R with $\tilde{H}(R)$ totally ordered and G (not only the positive cone of G) can be embedded into $\tilde{H}(R)$. The value group $G(A)$ is particularly useful in case A is a commutative valuation ring. The nonzero principal right ideals in a right chain ring R form a semigroup $H(R)$ under ideal multiplication only if R is right invariant. In the general case it is the semigroup $\tilde{H}(R)$ which takes the place of $H(R)$. Mathiak in [6] studies right and left chain domains with the help of a group that could be considered a generalization of $G(A)$. We found that in the case of one-sided conditions a generalization of $H(A)$, which will be a semigroup only, will be more natural.

2. Definition and preliminary results. We consider only rings