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ON THE BOUNDARY CURVES OF INCOMPRESSIBLE SURFACES

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Let K be a knot in S^3, and consider incompressible (in the stronger sense of π_1-injective), ∂-incompressible surfaces S in the exterior of K. A question which has been around for some time is whether the boundary-slope function $S \mapsto m_S/\ell_S$, where m_S and ℓ_S are the numbers of times each circle of ∂S wraps around K meridionally and longitudinally, takes on only finitely many values (for fixed K). This is known to be true for certain knots: torus knots, the figure-eight knot [4], 2-bridge knots [2], and alternating knots [3]. In this paper an affirmative answer is given not just for knot exteriors, but for all compact orientable irreducible 3-manifolds M with ∂M a torus. Further, we give a natural generalization to the case when ∂M is a union of tori.

To state this more general result it is convenient to use the projective lamination space $\mathscr{PL}(\partial M)$, defined in [4]. If ∂M is the union of tori $T_1, \cdots, T_n$, then $\mathscr{PL}(\partial M)$ is the join $\mathscr{PL}(T_1) * \cdots * \mathscr{PL}(T_n) = \boldsymbol{R}P^1 * \cdots * \boldsymbol{R}P^1$, a sphere S^{2n-1}. More concretely, suppose coordinates are chosen for each T_i. Then isotopy classes of finite systems of disjoint noncontractible simple closed curves on T_i are parametrized by the set $\boldsymbol{Z}^2/\pm$ of pairs $(a, b) \in \boldsymbol{Z}^2$, where (a, b) is identified with $(-a, -b)$. So systems on ∂M are parametrized by $(\boldsymbol{Z}^2/\pm)^n$. Restricting to nonempty systems and projectivising by identifying a system with any number of parallel copies of itself, yields $(\boldsymbol{Z}^2/\pm)^n - \{0\}/(v \sim \lambda v)$. This is the same as $(\boldsymbol{Q}^2/\pm)^n - \{0\}/(v \sim \lambda v)$. The natural completion of this space is $\mathscr{PL}(\partial M) = (\boldsymbol{R}^2/\pm)^n - \{0\}/(v \sim \lambda v)$, clearly a sphere of dimension $2n - 1$. (We shall not be concerned with the geometrical interpretation of the points added in forming this completion.) A change of coordinates for the T_i's produces a projective transformation of this S^{2n-1}, so $\mathscr{PL}(\partial M)$ has a natural projective structure. (For surfaces of higher genus, $\mathscr{PL}$ has only a natural piecewise projective structure.)

THEOREM. *Let M be orientable, compact, irreducible, with ∂M a union of n tori. Then the projective classes of curve systems in ∂M which bound incompressible, ∂-incompressible surfaces in M form a dense subset of a finite (projective) polyhedron in $\mathscr{PL}(\partial M) = S^{2n-1}$ of dimension less than n.*

COROLLARY. *If $\partial M = T^2$, there are just a finite number of slopes*