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PSEUDOCOMPACT GROUP TOPOLOGIES AND TOTALLY DENSE SUBGROUPS

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Throughout this synopsis all topologies are Hausdorff topological group topologies, and $\langle G, \mathscr{T} \rangle$ is assumed compact. The symbol w denotes weight. Definition. A subgroup H of $\langle G, \mathscr{T} \rangle$ is totally dense (in G) if $H \cap K$ is dense in K for every closed subgroup K of G. We prove these results.

If $\mathscr{T}' \supsetneqq \mathscr{T}$ with $\langle G, \mathscr{T}' \rangle$ pseudocompact, then not every $\mathscr{T}'$-closed subgroup of G is $\mathscr{T}$-closed. If $w(G, \mathscr{T}) > \omega$ with $\langle G, \mathscr{T} \rangle$ totally disconnected Abelian, then there is pseudocompact $\mathscr{T}' \supsetneqq \mathscr{T}$.

Not every infinite $\langle G, \mathscr{T} \rangle$ has a proper, totally dense subgroup. But (a) if $w\langle G, \mathscr{T} \rangle > \omega$ with $\langle G, \mathscr{T} \rangle$ connected Abelian, or (b) if $\langle G, \mathscr{T} \rangle$ is totally disconnected Abelian and in the dual group p-primary decomposition $\hat{G} = \oplus_p \hat{G}_p$ one has $|\hat{G}_p| > \omega$ for infinitely many primes p, then $\langle G, \mathscr{T} \rangle$ has a proper, totally dense, pseudocompact subgroup.

Let H be a totally dense subgroup of $\langle G, \mathscr{T} \rangle$. Then (a) $|G| \leqq 2^{|H|}$; (b) if G is Abelian then $|G| \leqq |H|^{\omega}$; (c) if G is connected Abelian then $|G| = |H|$; (d) if G is totally disconnected and H countably compact, then $G = H$; (e) there are examples with $\langle G, \mathscr{T} \rangle$ (totally disconnected) Abelian and $|H| < |G|$.

1. Introduction and motivation.

All the topological groups hypothesized in this paper are assumed to satisfy the T_0 separation property; as is well-known (see for example [18] (Theorem 8.4)), this guarantees that they are in fact completely regular, Hausdorff spaces, i.e., Tychonoff spaces.

The authors' interests in pseudocompact group topologies (defined below) arise, quite independently and to our surprise, from distinct and differing considerations not related at first glance: (a) as a natural class of spaces, considerably broader than the class of compact spaces, with fascinating topological features; and (b) as important tools in the development of topological Galois theory. As to (a) we note that major portions of [33], [20], [34] are devoted to questions concerning the existence and the size of dense pseudocompact subgroups of compact groups, while [7] shows that the product of pseudocompact groups is pseudocompact and that the

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