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EMBEDDING HOMOLOGY 3-SPHERES IN S^5

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The purpose of this note is to give a proof independent of high-dimensional surgery theory of the following embedding result:

THEOREM. Let Σ^3 be the homology 3-sphere resulting from a Dehn surgery of type $1/2a$ on a knot in S^3. Then Σ^3 smoothly embeds in S^5 with complement a homotopy circle.

This theorem illustrates the connection between two major areas of ignorance in low-dimensional topology. For instance, if the homology sphere Σ^3 bounds a contractible 4-manifold V^4, then, using the 5-dimensional Poincaré conjecture, we see that $\Sigma^3 \times 0 \hookrightarrow \Sigma^3 \times D^2 \cup V^4 \times S^1$ is a smooth embedding of Σ into S^5 with complement homotopy equivalent to a circle. Conversely, if Σ smoothly embeds in S^5 with $S^5 - \Sigma \simeq S^1$, and if the Browder-Levine fibering theorem [1] holds in dimension 5, then $S^5 - \Sigma^3 \times \mathring{D}^2$ fibers over S^1, and the fiber is necessarily contractible.

High dimensional surgery theory can be used to completely solve this problem. Given Σ^3, convert $\Sigma^3 \times T^2$ to $K \simeq S^3 \times T^2$ via surgery, with $\Sigma^3 \subset K$ (see [6]). By work of Kirby-Siebenmann, K is homeomorphic to $S^3 \times T^2$. Lifting to the universal cover, we get $\Sigma \subset S^3 \times \boldsymbol{R}^2 \subset S^5$, and we see that every homology 3-sphere topologically embeds in S^5 with complement a homotopy circle. However, if Σ has nontrivial Rochlin invariant, a standard argument shows that the embedding cannot be smooth or PL. (If it were smooth (PL), make the homotopy equivalence $f: S^5 - \Sigma^3 \times \mathring{D}^2 \to S^1$ transverse to a point $p \in S^1$. Then $f^{-1}(p)$ would be a smooth (PL) spin manifold V^4 with zero signature and $\partial V = \Sigma$, contradicting the fact that Σ has nontrivial Rochlin invariant.) If Σ has trivial Rochlin invariant, the argument in [8] shows that the embedding can be taken to be smooth or PL. (See [7] for a much deeper analysis of knotting of homology 3-spheres in S^5.) Nevertheless, it seems desirable to give a more elementary construction for these embeddings when possible. It would be nice if these methods, together with the Kirby-Rolfsen calculus for links in S^3, could provide the desired embeddings for all Σ^3 with zero Rochlin invariant.

This proof grew out of studying Fintushel and Pao's attempt [3] to show that Scharlemann's possibly exotic $S^3 \times S^1 \# S^2 \times S^2$ is standard [6]. The basic construction is from [3] and will be described below.