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ON ULTIMATELY NONEXPANSIVE SEMIGROUPS

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A semigroup G of continuous selfmappings of a metric space (X, d) is called ultimately nonexpansive if for every u, v in X and $\alpha>0$, there is an f in G such that for all g in G, $d(fg(u), fg(v)) \leqq (1+\alpha)d(u, v)$. It is shown that if G is an ultimately nonexpansive commutative semigroup of selfmappings, then G has a fixed point when any one of the following conditions is satisfied: (1) X is a reflexive Banach space and each orbit under G is precompact; (2) X is a finite dimensional Banach space and there is a point in X with a bounded orbit; (3) X is a reflexive, locally uniformly convex Banach space having a point with a precompact orbit.

1. Introduction. Let G be a family of selfmappings of a metric space (X, d) and suppose that G is a semigroup under composition. For any $x \in X$, $G(x) = \{g(x): g \in G\}$ is called the orbit of x under G and a point $z \in X$ such that $G(z) = \{z\}$ is called a fixed point of G. Fixed point properties of semigroups having various contractivity properties have been investigated by several writers (cf. [1], [3], [5], [6], [7], [8], [9], [10]). In [5], for example, an asymptotically nonexpansive semigroup G (see definition given below), satisfying certain conditions, was shown to have a fixed point if X is a finite dimensional Euclidean space and, for some $x \in X$, $\text{co}\, G(x)$ does not contain any affine flat of positive dimension. Similarly, in [8] a semigroup of selfmappings of a closed, bounded convex set in a uniformly convex Banach space—called eventually nonexpansive—was shown to have a fixed point provided certain additional hypotheses were satisfied.

In this paper we consider semigroups of mappings related to those of [8] but much more general; (e.g., members of the semigroup need not be lipschitzian).

DEFINITION. A semigroup G of continuous selfmappings of a metric space (X, d) is called ultimately nonexpansive if for every pair of points $u, v \in X$ and for every $\alpha > 0$, there is an $f \in G$ such that for all $g \in G$,

$$d(fg(u), fg(v)) \leqq (1 + \alpha)d(u, v) . \tag{1}$$

It is called asymptotically nonexpansive (cf. [5]) if (1) above is satisfied with $\alpha = 0$.

It is the purpose of this paper to show that some of the fixed point theory for semigroups of nonexpansive mappings carries over to this much larger class of mappings. In particular this is the case