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ON THE NUMBER OF REAL ROOTS OF POLYNOMIALS

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A general theorem concerning the structure of a certain real algebraic curve is proved. Consequences of this theorem include major extensions of classical theorems of Pólya and Schur on the reality of roots of polynomials.

1. Introduction and definitions. Throughout this paper, all polynomials are assumed to have real coefficients. In 1916, George Pólya wrote a paper [6] in which he considered two polynomials, $f(x) = \sum_{i=0}^{m} a_i x^i$, $a_m \neq 0$, and $h(x) = \sum_{i=0}^{n} b_i x^i$, $b_n \neq 0$, where $n \geqq m$, both polynomials have only real roots and the roots of h are all negative. Under these hypotheses, he proved the following theorem.

THEOREM 1.1 (Pólya [6]). *The real algebraic curve* $F(x, y) \equiv b_0 f(y) + b_1 x f'(y) + \cdots + b_m x^m f^{(m)}(y) = 0$ *has* m *intersection points with each line* $sx - ty + u = 0$, *where* $s \geqq 0$, $t \geqq 0$, $s + t > 0$ *and* u *is real.*

Pólya then noted that this theorem gives a unified proof of three important special cases regarding composite polynomials:

(1.2)(a). Setting $x = 1$ gives a special case of the Hermite-Poulain theorem [5, Satz 3.1].

(1.2)(b). Setting $y = 0$ gives a theorem of Schur [5, Satz 7.4].

(1.2)(c). Setting $x = y$ gives a theorem of Schur and Pólya [7, p. 107].

Our main theorem, proved in § 2 establishes some of the properties of $F(x, y) = 0$ when we drop all restrictions on f and require only that all the roots of h be real (of arbitrary sign). In the last section, we apply this theorem to obtain an extension of Theorem 1.1 which states that, for h restricted as in Pólya's theorem, there are at least as many intersection points as the number of real roots of f. As corollaries, we then obtain the full strength of the Hermite-Poulain theorem and extensions of the theorems of Pólya and Schur to arbitrary polynomials f. Pólya states that his theorem is one of the most general known theorems on the reality of roots of polynomials. We believe this statement is still true