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ON PROPERTY (Q) AND OTHER SEMICONTINUITY PROPERTIES OF MULTIFUNCTIONS

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Different upper semicontinuity properties of multifunctions in general topological spaces are presented and their interrelationships are expounded in detail. In particular criteria are given for Cesari's property (Q) for multifunctions $f: X \to E$ where X is a general topological space and E is a locally convex space. Among them are that either f is mild upper semicontinuous, or f is maximal monotone.

Introduction. In addition to their intrinsic mathematical interest, the study of upper semicontinuity properties of multifunctions has been motivated by numerous applications in different fields, for instance, in optimal control problems (Cesari [5], [6], [7]), in mathematical programming (Zangwill [20]), and in nonlinear functional analysis (Brézis [2]).

Several concepts of upper semicontinuity (u.s.c.) have been introduced in the past at various levels of generality. In the present paper we present these concepts in general topological spaces: the closed graph property, the u.s.c. property, and property (Q). Moreover, for the sake of comparison, we also introduce the concept of mild u.s.c. The comparison of these concepts at this level of generality does not seem to have been done before. Also, we prove here a number of implications which seem to be new. For instance, we prove in § 2 that mild u.s.c. implies property (Q) and that mild u.s.c. also implies the closed graph property. The latter is a slightly more general statement than the essentially known result that u.s.c. implies the closed graph property. In particular we prove in § 6 a result which seems to be of some relevance, that is, a maximal monotone multifunction in any locally convex space E satisfies all the upper semicontinuity properties discussed in this paper if it is locally bounded. Thus a maximal monotone multifunction necessarily has property (Q) if it is locally bounded. In the case that E is a Frechét space, maximal monotonicity alone is sufficient to imply property (Q). This last result extends a previous one of Suryanarayana [17].

All these results seem to indicate how property (Q) appears to be a generalization of a number of different concepts introduced in different fields. The following diagram summarizes some of the interrelationships among all these concepts.

Let $f: X \to Y$ be a multifunction concerning two topological