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AN EXTENSION OF SION'S MINIMAX THEOREM WITH AN APPLICATION TO A METHOD FOR CONSTRAINED GAMES

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Sion's minimax theorem is extended for noncompact sets, and for certain two-person zero-sum games on constrained sets a sequential unconstrained solution method is given.

I. Introduction. It is an important question in two-person zero-sum games, whether there exists a saddle point strategy, and if so, how it is to be computed. Existence theorems are known almost only for the case that the sets of strategies are compact. Often these sets are given by numerically complicated conditions and because of the necessity to consider the boundary of the constraint region you cannot apply analytical methods.

First we extend Sion's minimax theorem [7] for noncompact sets. With it we then give a solution method for a frequently occurring type of games over constrained sets. This method approximates a solution from the interior of the admissible sets and makes it possible to apply analytical methods like those for the whole spaces. It can be regarded as an extension of the widely used Interior Penalty Method of Nonlinear Programming to saddle point problems.

II. A minimax theorem for noncompact sets. Let X and Y be not empty subsets of real linear topological Hausdorff spaces $\mathscr{X}$ and $\mathscr{Y}$, respectively, and let $\mathbf{R}$ denote the real numbers.

DEFINITION 1.
(a) A function $f: X \to \boldsymbol{R}$ is called
(i) *inf-compact* if $\{x \mid x \in X,\ f(x) \leqq a\}$, $a \in \boldsymbol{R}$, is compact,
(ii) *sup-compact* if $\{x \mid x \in X,\ f(x) \geqq a\}$, $a \in \boldsymbol{R}$, is compact.
(b) A function $f: X \times Y \to \boldsymbol{R}$ is called (x_1, y_1)-*sup inf-compact*, for a fixed $(x_1, y_1) \in X \times Y$, if $f(x_1, \cdot)$ is inf-compact and $f(\cdot, y_1)$ is sup-compact.

If $f: X \times Y \to \boldsymbol{R}$ is u.s.c.-l.s.c., i.e., $f(x, y)$ is upper semi-continuous in x for each $y \in Y$ and lower semi-continuous in y for each $x \in X$, and X and Y are compact sets, then $f(x, y)$ is (x_1, y_1)-sup inf-compact for all $(x_1, y_1) \in X \times Y$. Thus the following theorem generalizes Theorem 3.4 of Sion [7].

THEOREM 1. *Let X and Y be convex sets, and $f: X \times Y \to \boldsymbol{R}$ an*