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LIFTING GROUP HOMOMORPHISMS

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If a knot K has Alexander polynomial different from 1, then its knot group, G maps onto some metacyclic group, $Z_r \rtimes Z_p$. We show that in that case, it also has a homomorphism onto a split extension of a free abelian group of rank $p-1$ by $Z_r \rtimes Z_p$, and hence also onto a split extension of a direct sum of $p-1$ cyclic groups of order s by the metacyclic group. In many cases, (such as if s is coprime with p), this group can be specified exactly. Otherwise there are a finite number of possibilities. A special case is Perko's result that a homomorphism of a knot group onto $S_3 = Z_2 \rtimes Z_3$ lifts to $S_4 = Z_2 \rtimes Z_3 \rtimes (Z_2 \oplus Z_2)$.

As an application we obtain information about the derived series of G.

In a final section it is shown how to associate a rational polynomial invariant to every metacyclic representation.

1. Lifting metacyclic representations. Let p be a prime, r a divisor of $p-1$ and β a primitive rth root modulo p. Let $E = Z_r \rtimes Z_p = \langle Y, S\colon Y^r = S^p = 1, Y^{-1}SY = S^\beta \rangle$. Up to isomorphism, the group is independent of β. Let G be the knot group of a knot, K, in the 3-sphere, S^3, and let ϕ be a homomorphism of G onto E which takes a meridian, m, of K to Y^aS^b. Then g.c.d. $(a, r) = 1$, since G is generated by conjugates of m. Setting $X = Y^aS^b$ and eliminating Y, we obtain a presentation

$$E = \langle X, S\colon X^r = S^p = 1,\ X^{-1}SX = S^\alpha \rangle,$$

where $\alpha = \beta^a$ and $m\phi = X$. We describe this situation by saying that ϕ *maps G onto $E(\alpha)$*, meaning that $(m\phi)^{-1}Sm\phi = S^\alpha$. The following condition is well known: [**6, 3**]

(1.1) *G maps onto $E(\alpha)$ if and only if p divides $\Delta(\alpha)$* where Δ is the Alexander polynomial of K.

We assume throughout this paper that ϕ maps G onto $E(\alpha)$.

Let η be a primitive pth root of unity, and Q the rational numbers. Then $Q(\eta)$ can be given the structure of an E-module by

$$V^S = V.\eta \quad \textit{and} \quad V^X = V\sigma \tag{1.2}$$

for $V \in Q(\eta)$, where σ is the Galois automorphism of $Q(\eta)$ determined by $\eta\sigma = \eta^\alpha$. (Module action is denoted by writing the element of E as a superscript.)