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THE EXISTENCE OF STRONG LIFTINGS FOR TOTALLY ORDERED MEASURE SPACES

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Let X be a totally ordered space, μ a finite Borel measure on X with full support, and $\mathfrak{F}$ the σ-algebra of all μ^*-measurable subsets of X. Then there exists a lifting $\rho\colon \mathfrak{F} \to \mathfrak{F}$ which satisfies $U \subset \rho(U)$ for every open subset U of X.

Assume that X is a topological space and μ a finite, Borel measure on X with full support. We are interested in finding conditions for the topology of X, which insure the existence of strong liftings for the associated topological measure space. In **[8]** Losert has given an example showing that this is not always possible even if X is compact. On the other hand Graf has proved that strong liftings always exist for measures on second countable spaces **[6]**. Other positive results on the existence of strong liftings are given in **[2]** and **[4]**.

In this paper we show that every totally ordered measure space admits a strong lifting. Moreover we prove that if μ is a Radon, non-atomic measure on a totally ordered space then every lifting of the associated measure space is almost strong, if and only if, the set of all two sided limit points of the support of μ is μ^*-measurable with full measure.

1. Preliminaries and notation. Throughout X will be a set and "$\leq$" a total order on X. If x, y are two points of X, let $x < y$ means that $x \leq y$ and $x \neq y$. Let $(-\infty, x) = \{z \in X\colon z < x\}$, $(x, +\infty) = \{z \in X\colon x < z\}$ and $(x, y) = (-\infty, y) \cap (x, +\infty)$. Assume that $Y \subset X$ and $y \in Y$. We say that y is a left limit point of Y if $(-\infty, y) \cap Y \neq \varnothing$ and there is no z in Y such that $z < y$ and $(z, y) \cap Y = \varnothing$. Analogous is the definition of the right limit point. A point y in Y is said to be a two sided limit point of Y if it is both a left and right limit point of Y.

We say that $(X, \leq)$ is a totally ordered (topological) space if its topology is generated by all the intervals of the form $(-\infty, x)$, $(x, +\infty)$. By a measure on X we mean a finite, non-negative, countably additive set function defined on the Borel sets of X. A measure μ on X is said to be Radon if it is inner approximated by the compact sets of X. The support S_μ of a measure μ on X is defined by

$$S_\mu = \bigcap \{F\colon F \text{ closed subset of } X \text{ such that } \mu(F) = \mu(X)\}.$$