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THERE ARE NO PHANTOM COHOMOLOGY OPERATIONS IN K-THEORY

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Let h_1^*, h_2^* be cohomology theories defined on the category of finite CW-complexes, and suppose that h_1^*(point), h_2^*(point) are both countable. Then by Brown's [5] Representability Theorem, there are Ω-spectra $\mathbf{Z}_1$, $\mathbf{Z}_2$ such that $h_i^*(X) = [X, \mathbf{Z}_i]_*$, the graded group of homotopy classes of maps of X into the terms of the spectrum. If we exercise some care in the choice of $\mathbf{Z}_1$, we shall see that every stable cohomology operation φ: $h_1^*(X) \to h_2^*(X)$ defined for X finite extends to a map φ: $\mathbf{Z}_1 \to \mathbf{Z}_2$ of spectra. We shall examine the question: How many choices, up to homotopy, are there for φ, given φ? As an intermediate question, we shall also investigate: How many extensions are there to infinite CW-complexes are there of φ?

In the case when h_1^*, h_2^* are the connected forms of K-theory (real, symplectic, or complex), we shall show that every cohomology operation extends uniquely from finite complexes to all complexes and that the spectral homotopy class of the representing map is unique. Since a cohomology class which vanishes on all finite subcomplexes of a complex is called a phantom class, we shall call a cohomology operation which vanishes on all finite complexes a phantom operation. We shall call a spectral map of Ω-spectra a completely phantom cohomology operation if it vanishes on all CW-complexes. Our main theorem is as follows.

THEOREM 1. *There are no phantom or completely phantom cohomology operations other than the zero operations between the connected forms of complex, real, or symplectic K-theory, and every stable cohomology operation defined between these theories on finite complexes is represented by a spectral map which is unique up to homotopy.*

In the course of the proof of Theorem 1, we prove another theorem. Let E_{**}^r be the classical to general cohomology spectral sequence (known to the K-theorists as the Atiyah-Hirzebruch spectral sequence and to homotopy theorists by many names).

THEOREM 2. *Let E_{**}^r be the classical to general cohomology spectral sequence for computing the stable cohomology operations from connected complex K-theory to itself. Then $E_{p,q}^r$ is finite if p is odd, zero if q is odd or*