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BERNSTEIN-LIKE POLYNOMIAL APPROXIMATION IN HIGHER DIMENSIONS

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Let $C(K)$ be the Banach space of continuous, real-valued functions defined on a compact, Hausorff space, K, let $\mathcal{P} = \mathcal{P}(K)$ be the positive linear forms, P, defined on $C(K)$, for which $Pf \leq \sup f(k)$ $(k \in K)$, $f \in C(K)$, and endow $\mathcal{P}(K)$ with the weak-star topology in which it is, of course, compact. (As is well known, $\mathcal{P}$ can be identified with the set of countably additive probability measures defined on the Baire subsets of K.) Let P^∞ be the power probability on K^∞, the product of denumerable number of copies of K. Then, for each $f \in C(K^\infty)$, the integral of f with respect to P^∞ is plainly continuous in P. As Theorem 2 below states, there are no ohter continuous real-valued functions of P. The proof of this assertion requires a generalization of Bernstein's version of the celebrated polynomial approximation theorem of Weierstrass, which generalization is provided by Theorem 1.

The two theorems are numbered in their logical order but stated in the order of simplicity of formulation.

THEOREM 2. *For every $g \in C(\mathcal{P}(K))$, there is an $f \in C(K^\infty)$ such that, for all $P \in \mathcal{P}(K)$,*

$$\int f\, dP^\infty = g(P). \tag{1}$$

It is convenient to reformulate Theorem 2 in terms of an operator T mapping $C(K^\infty)$ into $C(\mathcal{P}(K))$ defined, thus.

$$(Tf)P = \int f\, dP^\infty, \qquad f \in C(K^\infty), P \in \mathcal{P}(K). \tag{2}$$

Reformulated, Theorem 2, states that the operator T is surjective, that is, onto $C(\mathcal{P})$. A short digression explains the origin of this theorem.

Suppose Q is a probabilistic mixture of power probabilities,

$$Q = \int P^\infty \mu(dP) \tag{3}$$

for some probability μ on $\mathcal{P}$, or, more fully,

$$Qf = \int (Tf)(P)\, \mu(dP), \qquad f \in C(K). \tag{4}$$