PACIFIC JOURNAL OF MATHEMATICS
Vol 111, No 1, 1984

SOME CONDITIONS ON THE HOMOLOGY GROUPS OF THE KOSZUL COMPLEX

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In this paper we introduce the concept of a (d, i)-sequence ($d, i \in N$) in a commutative ring A, noetherian and with identity (cf. Def. 1.1). Let $K(\underline{z}, A)$ be the Koszul complex on A, with respect to the sequence $\underline{z} = z_1, \dots, z_n$: the concept of a (d, i)-sequence is expressed in terms of the structure of $H_i(K(\underline{z}, A))$; in particular, it turns out that $\underline{z}$ is an (n, i)-sequence iff $H_i(K(\underline{z}, A)) = 0$, and such a condition implies $\underline{z}$ is a (d, i)-sequence for any $d \leq n$. If $\bar{z}_1, \dots, \bar{z}_h$ is a (d, i)-sequence in ${}_h\bar{A} = A/(z_{h+1}, \dots, z_n)$, $d \leq h \leq n$, then $\underline{z}$ is seen to be a (d, i)-sequence in A; so, in particular, if $H_i(K(\underline{\bar{z}}; {}_d\bar{A})) = 0$ in ${}_d\bar{A}$, then $\underline{z}$ is a (d, i)-sequence. Moreover, for $i = 1$, the two conditions are equivalent, so that *$\underline{z}$ is a $(d, 1)$-sequence* means precisely that $\bar{z}_1, \dots, \bar{z}_d$ is regular in ${}_d\bar{A}$. For $i > 1$, examples show that *$\underline{z}$ is a (d, i)-sequence* is a condition strictly weaker than *$\bar{z}_1, \dots, \bar{z}_h$ is a (d, i)-sequence in ${}_h\bar{A}$*, and we investigate the relationship between those two properties. In fact, their equivalence allows us to read the depth of a quotient ring $A/(z_{h+1}, \dots, z_n)$ in terms of the Koszul complex $K(\underline{z}; A)$ and implies, for (d, i)-sequences, properties which are a natural generalization of *good properties* satisfied by regular sequences, such as the depth-sensitivity of the Koszul complex. A characteristic condition for their equivalence is a kind of weak surjectivity of a natural map acting between $\operatorname{syz}^{i+1}(K(\underline{z}; A))$ and $\operatorname{syz}^{i+1}(K(\underline{\bar{z}}; {}_h\bar{A}))$.

From an algebraic form of that weak surjectivity we get some sufficient conditions, in terms of weak regularity of the sequence $z_{h+1}, \dots, z_n$. For instance, if $z_{h+1}, \dots, z_n$ is a d-sequence, or a relative regular sequence, or less, if $z_{h+1}, \dots, z_n$ is a relative regular A-sequence with respect to a convenient set of ideals, then *$\underline{z}$ is a (d, i)-sequence in A* implies *$\bar{z}_1, \dots, \bar{z}_h$ is a (d, i)-sequence in ${}_h\bar{A}$*.

Moreover, if $\underline{z}$ is a (d, i)-sequence and $z_{d+1}, \dots, z_n$ is a regular sequence, then $H_i(K(\underline{z}; A)) = 0$, while this vanishing implies that it is possible to find $x_i, \dots, x_n$ in $I = (z_1, \dots, z_n)$ such that $z_1, \dots, z_{i-1}, x_i, \dots, x_n$ is a (d, i)-sequence and $x_{d+1}, \dots, x_n$ is a regular sequence.

In the last section we give an interpretation of our results in terms of the behaviour of some systems of linear equations.

N. 1. Let A be a noetherian ring (with 1) and $\underline{z} = z_1, \dots, z_n$ a sequence of elements of A such that $(z_1, \dots, z_n)A \neq A$. We denote by $K(\underline{z}; A)$ the Koszul complex with respect to $\underline{z}$, i.e. the differential graded algebra (DGA for short) (cf. **[G-L]** cap. I for a definition)

$$0 \to \wedge^n A^n \xrightarrow{d_n} \wedge^{n-1} A^n \to \cdots \to \wedge^2 A^n \xrightarrow{d_2} A^n \xrightarrow{d_1} A \xrightarrow{d_0} sA/(\underline{z})A \to 0$$