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CONTRIBUTIONS TO HILBERT'S EIGHTEENTH PROBLEM

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In the second part of his eighteenth problem Hilbert formulated: "A fundamental region of each group of Euclidean motions together with all its congruent copies evidently gives rise to a covering of the space without gaps. The question arises as to the existence of such polyhedra which cannot be fundamental regions of any group of motions, but nevertheless furnish such a covering of the total space by congruent reiteration." Following ideas of Heesch this question of Hilbert's will be analysed in detail in this article, restricting to the case of two dimensions — the Euclidean plane E^2.

A. Introduction. After some necessary terminology has been introduced the above question is subdivided into two special questions: (1) Do there exist tiles B_i forming homogeneous tilings but having only non-trivial stabilizers? (2) Do there exist tiles B_i, which tile the E^2 but cannot form a homogeneous tiling? Positive answers to both special questions are obtained for the case of coverings by non-compact tiles, in terms of construction rules for special types of regions in the form of strips. These complete the list of previously published examples. Subsequently, a new method is developed. By systematic subdivision of the above strips, new types (both non-compact and compact) will be constructed also yielding a positive answer to Hilbert's original question.

A positive answer of a special kind can be given immediately for the case of a fundamental region B of some discrete group $\mathfrak{G}$, if $B \neq \bar{B}$ (closure of B): For this case the polyhedron $\bar{B}$ can fill space under the "sharply transitive" action of $\mathfrak{G}$, and yet is not a fundamental region of $\mathfrak{G}$, since its boundaries contain equivalent points.

K. Reinhardt in 1928 was the first to answer the Hilbert question affirmatively by giving a 3-dimensional multiconnected polyhedron. H. Heesch gave a similar answer in 1935 for E^2 by constructing a special decagon (cf. [**3**]). Further results given by Heesch are cited at the end of §C, after the detailed analysis has been carried out. For a survey of the development in the Hilbert problem and its generalizations we refer to [**2**].

B. Definitions.

DEFINITION 1. A *tiling* Π of E^2 is a covering of E^2, without overlap, by any set of compact or non-compact topological disks (called tiles). In