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ON THE REVERSE WEAK TYPE INEQUALITY FOR THE HARDY MAXIMAL FUNCTION AND THE WEIGHTED CLASSES $L(\log L)^k$

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Muckenhoupt has given a necessary and sufficient condition to be satisfied by the weight functions U and V in order that the Hardy-Littlewood maximal function Mf should satisfy a weighted weak type (1, 1) inequality. In this note, conditions on the weight functions U and V are given in order that the sense of this inequality may be reversed. This is then applied to give conditions which ensure that the integrability of Mf with respect to a weight implies that f belongs to a weighted Zygmund class $L\log L$, thus extending a result of Stein. Analogous results related to the strong maximal function and the classes $L(\log L)^k$ are also given. These extend certain results of Favo, Gatto and Gutiérrez.

If f is locally integrable on R^n, the Hardy-Littlewood maximal function Mf is defined by

$$(Mf)(x) = \sup|Q|^{-1}\int_Q |f|$$

where the supremum is taken over all cubes Q containing x. Here and henceforth, by "cube" we shall always mean "cube with sides parallel to the co-ordinate axis". As usual, $|E|$ denotes the Lebesgue measure of the measurable set E, and more generally if $U(x) \geq 0$ is defined on E we write $|E|_U = \int_E U(x)\,dx$. If Q is a given cube, RQ denotes the cube concentric with Q but with side R times as long. Q_0 will denote a fixed but arbitrary cube in R^n. Our first result is the following theorem.

THEOREM 1. *Suppose the non-negative weight functions U and V are defined on Q_0. If there is a constant C depending only on U and V such that*

$$(1) \qquad \frac{|Q|_U}{|Q|} \geq C \operatorname*{ess\,sup}_{x\in Q} V(x) \quad \textit{for all cubes } Q \subset Q_0,$$

then

$$(2) \qquad |\{x \in Q_0 : (Mf)(x) > \lambda\}|_U \geq C2^{-n}\lambda^{-1}\int_{\{x:f(x)>\lambda\}} f(x)V(x)\,dx$$