PACIFIC JOURNAL OF MATHEMATICS
Vol. 113, No 1, 1984

EXPONENTIAL RINGS, EXPONENTIAL POLYNOMIALS AND EXPONENTIAL FUNCTIONS

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In this paper we define the category of exponential rings and develop some of its basic properties.

Introduction. An exponential ring, or E-ring for short, is a pair (R, E) with R a ring—in this paper always commutative with 1—and E a morphism of the additive group of R into the multiplicative group of units of R, that is, $E(x+y) = E(x)E(y)$ for all x, y in R, and $E(0) = 1$. Examples are $(\mathbf{R}, a^x)$, a any positive real, and $(\mathbf{C}, e^x)$. Of course, any ring R can be expanded to an E-ring (R, E) by putting $E(x) = 1$ for all x; such E-rings will be called trivial. Ken Manders observed that an E-ring whose underlying ring has no nilpotents $\neq 0$ and has characteristic a prime $p > 0$ is trivial: in such a ring each x satisfies $1 = E(0) = E(px) = E(x)^p$, so $(E(x) - 1)^p = 0$, which implies $E(x) = 1$.

Related notions of exponential ring have been considered by M. Beeson, by B. Dahn and Wolter, and by A. Wilkie, all in connection with the longstanding open problem of A. Tarski on the decidability of the field of reals with exponentiation. An effective positive solution to this problem seems unlikely without major advances in transcendental number theory: such a solution would give us a decision method to answer any question: is $e^e = p/q$?, where p, q are positive integers. Of course there is such a decision method, but, as we don't know yet whether e^e is rational, we don't know how it works.

Now in mathematical practice it is less the *effectiveness* of Tarski's decision method for the real field which matters—though this aspect is interesting—but rather the information the method provides on the algebraic-topological nature of the definable sets in $\mathbf{R}^m$, and on the asymptotic behavior of definable functions. For example in semi-algebraic and real algebraic geometry this use is formalized in the Tarski-Seidenberg theorem (in an inconstructive version) and in a result like the finiteness of the number of connected components of a semi-algebraic set.

Parts of this use of Tarski's work on the elementary theory of the reals offer more hope of being generalized to the E-ring $(\mathbf{R}, e^x)$. The following