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COBORDISM OBSTRUCTIONS TO FIBERING MANIFOLDS OVER SPHERES

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We consider here the problem introduced by Conner and Floyd of determining necessary and sufficient conditions for a manifold M to be cobordant to a bundle over a given sphere S^k. Two recent studies by D. F. X. O'Reilly [7] and A. Didierjean [4] presented obstructions to fibering manifolds over spheres in terms of the "top" Stiefel-Whitney classes of M. While these conditions were shown by Conner and Floyd [3] and R. L. W. Brown [2] to be sufficient when restricted to the cases of fiberings over S^1 and S^2, they are not at all sufficient for guaranteeing the fibering of a cobordism class over a sphere of any higher dimension. This is shown in O'Reilly's study of fiberings over the 4-sphere.

In this paper we exhibit an obstruction to fibering a manifold over a sphere that extends the obstructions mentioned above. We then essentially answer all open questions but one regarding the problem of which cobordism classes can be represented by a bundle over S^4.

1. Introduction. In [3], Conner and Floyd introduced the problem of determining which cobordism classes in $\mathfrak{N}_*$ could be represented by a manifold fibered (smoothly) over a given sphere. They showed that if $w_i(M)$ denotes the ith Stiefel-Whitney class of a manifold M, then a class $\omega \in \mathfrak{N}_n$ fibers over S^1 (i.e. ω contains a representative that fibers over S^1) if and only if the Stiefel-Whitney number $w_n(\omega) = 0$. Extending the study, R. L. W.Brown [2] showed that a class $\omega \in \mathfrak{N}_n$ fibers over S^2 if and only if the Stiefel-Whitney number $w_n(\omega) = 0$ if n is even and $w_2 w_{n-2}(\omega) = 0$ if n is odd. Subsequent investigations have considered fiberings over a variety of manifolds, yielding complete solutions in a number of cases. But R. E. Stong [8] observed that spheres, aside from having a natural importance, actually play a key role here. He showed

(1.1) If a class $\omega \in \mathfrak{N}_n$ fibers over S^k, then ω fibers over any manifold N^q with $q \leq k$.

At present, for fiberings over spheres of any higher dimension than two, there are only partial results. In solving the S^2 problem, Brown exhibited a general necessary condition [2, Prop. 2.1] for a manifold to fiber over a sphere S^k that is based on the Brown-Peterson relations among characteristic classes. Using it, he derived the Stiefel-Whitney number obstructions mentioned above. D. F. X. O'Reilly [7] (see also A.