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EXTENDING LEBESGUE MEASURE BY INFINITELY MANY SETS

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Consider the following question: under what conditions on a collection of subsets of the unit interval can the existence of an extension of Lebesgue measure defined on each element of the collection be guaranteed? The main purpose of this paper is to find conditions on the cardinality of the collection whose sufficiency can be shown consistent without the use of large cardinals. For example, if ZFC is consistent so is ZFC + "Lebesgue measure can be extended to any countable collection of sets".

The results of this paper complement work of earlier researchers. Banach and Kuratowski showed that assuming the continuum hypothesis there is a countable collection of sets of reals for which no extension exists. Solovay proved that an extension of Lebesgue measure to all sets is equiconsistent with the existence of a measurable cardinal.

1. Definitions and notation. Almost all definitions and notations are standard but I will make a few remarks in hopes of avoiding any problems.

${}^{X}Y$ denotes the collection of functions from X into Y. The concatenation of two sequences s and t is written as $s \frown t$. $\langle a_i : i \in I\rangle$ denotes the function $i \mapsto a_i$ with domain I. CBA is short for complete Boolean algebra. If T is a tree then subtrees are always intended to be closed downward (trees grow upward). A tree T is $< \rho$ branching for a cardinal ρ if the collection of immediate successors of an element in T always has cardinality less than ρ. If $\mathscr{C}$ is a complete subalgebra of a CBA $\mathscr{B}$ and G is a V-generic ultrafilter on $\mathscr{C}$, I use $\mathscr{B}/G$ for the quotient of $\mathscr{B}$ by the ideal generated in $\mathscr{B}$ by the dual of G.

Jech [**4**] is a good reference for the set theoretic aspects of this paper.

Given a set X and a cardinal ρ, a ρ-algebra on X is an algebra of subsets of X which is closed under unions of size less than ρ. σ-algebra means the same as ω_1-algebra as usual. Measures are only assumed to be finitely additive and with a domain consisting of an algebra of sets (not necessarily a σ-algebra). Suppose $\mathscr{A}$ is an algebra of subsets of X and ν is a measure defined on $\mathscr{A}$. I will always tacitly assume that the measure of each element in $\mathscr{A}$ is in $[0,1]$ and $\nu(X) = 1$. ν is ρ-additive if whenever $A_i \in \mathscr{A}$ for $i \in I$ are pairwise disjoint with $\bigcup_{i \in I} A_i = X$ and $|I| < \rho$ then $\sum_{i \in I} \nu(A_i) = 1$.

Halmos [**3**] may be used for the basic facts on product measures.