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A NEW KIND OF EIGENFUNCTION EXPANSIONS ON GROUPS

HORST LEPTIN

Let G be a locally compact group, $C_\infty(G)$ the Banach algebra of C-valued continuous functions on G vanishing at infinity, and let $\mathscr{Q}$ be a translation-invariant dense $*$-subalgebra. We assume that $\mathscr{Q}$ has its own norm, such that it is a Banach G-algebra with involution. Then the twisted convolution algebra $\mathscr{L}= L^1(G, \mathscr{Q})$ is simple and symmetric and there exists — up to unitary equivalence — exactly one irreducible $*$-representation λ, mapping $\mathscr{L}$ into the compact operators of $L^2(G)$. Thus for hermitian $f \in \mathscr{L}$ one has the canonical spectral decomposition $\lambda(f) = \Sigma_j \alpha_j E_j$ with $\{\alpha_j\} = \operatorname{Spec}\lambda(f) = \operatorname{Spec}_{\mathscr{L}}(f)$, E_j finite-dimensional projections in $L^2(G)$. It turns out that $E_j = \lambda(e_j)$ for idempotent $e_j \in \mathscr{L}$, hence every hermitian $f \in \mathscr{L}$ defines uniquely a Fourier series $\Sigma \alpha_j e_j$ in $\mathscr{L}$. Different convergence properties of such expansions are studied.

The main result states that for "radial functions" f the eigenfunctions e_j span a maximal commutative subalgebras of $\mathscr{L}$ and that there exists a summation method for these f, generalizing the Fejer kernel for periodic functions. More precisely: There exists a bounded approximate identity for $\mathscr{L}$, consisting of finite linear combinations of the e_j. Applications are given to algebras $L^1(N)$ for nilpotent Lie groups N, in particular all such N are determined, on which a compact abelian group K acts such that the subalgebra $L^1_K(N)$ of radial (i.e. K-invariant) functions is commutative.

Let G be a locally compact group with a liminal C^*-group-algebra. Then every irreducible unitary representation π of G, resp. of $L^1(G)$, maps $L^2(G)$ into the compact operators of the Hilbert space $\mathscr{H}(\pi)$. Thus for a hermitian function $f \in L^1(G)$ the operator $\pi(f)$ has a spectral decomposition $\pi(f) = \Sigma_0^\infty \alpha_j E_j$ with orthogonal minimal projectors E_j of $\mathscr{H}(\pi)$. It can happen that also the E_i are in the image $\pi(L^1)$, i.e. $E_i = \pi(e_i)$ with $e_i \in L^1(G)$. In this case it is reasonable to say that

$$(1) \qquad f \sim \sum_{i=0}^{\infty} \alpha_i e_i \quad (\operatorname{mod} \ker \pi)$$

is an *eigenfunction expansion* modulo π of f, and to ask in which sense the series $\Sigma \alpha_j e_j$ converges to f.

To give an example let us take for G the Mackey group of $H = \mathbf{T} \times \mathbf{Z}$, $\mathbf{T}$ the circle group, with respect to the cocycle $c((\zeta, m), (\vartheta, n)) = \vartheta^m$. Thus $G = H \times \mathbf{T}$ with product $(x, \alpha)(y, \beta) = (xy, c(x, y)\alpha\beta)$. The group G is