PACIFIC JOURNAL OF MATHEMATICS
Vol 116, No 2, 1985

COUNTING SUBGROUPS AND TOPOLOGICAL GROUP TOPOLOGIES

SHIFERAW BERHANU, W. W. COMFORT AND J. D. REID

Let G be an Abelian group with $|G| = \alpha \geq \omega$, $\mathscr{S}(G)$ the set of subgroups of G, $\mathscr{B}$ the set of totally bounded topological group topologies on G, $\mathscr{M}(\gamma)$ the set of topological group topologies $\mathscr{T}$ for which the character (= local weight) of $\langle G, \mathscr{T}\rangle$ is equal to $\gamma \geq \omega$, and $\mathscr{B}(\gamma) = \mathscr{B} \cap \mathscr{M}(\gamma)$. We prove algebraic results and topological results, as follows.

Algebra. Either $|\mathscr{S}(G)| = 2^\alpha$ or $|\mathscr{S}(G)| = \alpha$. If $|\mathscr{S}(G)| = \alpha$ then $\alpha = \omega$. We describe and characterize those (countable) G such that $|\mathscr{S}(G)| = \omega$, and we give several examples.

Topology. If $\gamma < \log(\alpha)$ or $\gamma > 2^\alpha$, then $\mathscr{B}(\gamma) = \varnothing$; otherwise $|\mathscr{B}(\gamma)| = 2^{\alpha\cdot\gamma}$. If $\gamma > 2^\alpha$ then $\mathscr{M}(\gamma) = \varnothing$; if $\log(\alpha) < \gamma \leq 2^\alpha$ then $|\mathscr{M}(\gamma)| = 2^{\alpha\ \gamma}$; and if $\omega \leq \gamma \leq \alpha$ then $|\mathscr{M}(\gamma)| = 2^\alpha$.

0. Introduction and motivation. As a reading of the Synopsis may suggest, this work originated with the authors' interest in the following questions: Given an infinite Abelian group, how many topological group topologies does G possess? Of these, how many may be chosen pairwise non-homeomorphic? How many metrizable? How many totally bounded? How many totally bounded and metrizable? We approached the latter questions through a result from **[6]** which gives a one-to-one order-preserving correspondence between the set $\mathscr{B}(G)$ of totally bounded topological group topologies for G and the set of point-separating subgroups of the homomorphism group $\mathrm{Hom}(G, \mathbf{T})$. Thus it became natural—indeed necessary—to count the number of subgroups of a group of the form $\mathrm{Hom}(G, \mathbf{T})$. In §§1 and 2, which we believe have algebraic interest quite independent of their topological roots, we do a bit more: We show that every uncountable Abelian group G has $2^{|G|}$ subgroups, and we describe in some detail the fine algebraic structure of what we call ω-groups. These are by definition the (necessarily countable) Abelian groups G with fewer then $2^{|G|}$ subgroups; we show that each ω-group has exactly ω-many subgroups, and we describe the relationship between the ω-groups and the so-called q.d. groups of Beaumont and Pierce **[2]**.

The algebraic analysis of §1, together with the result cited from **[6]**, allows us to describe some gross features of the partially ordered sets $\mathscr{B}(G)$. Here our work is sufficiently coarse that the various cardinal