PACIFIC JOURNAL OF MATHEMATICS
Vol 118, No 2, 1985

EQUATIONS IN PRIME POWERS

D. ESTES, R. GURALNICK, M. SCHACHER AND E. STRAUS(*)

Equations of form $p^a = (q^n - 1)/(q - 1)$ are considered where p is prime, q a prime power, and $n \geq 3$. These equations occur in group theory and in number theory in an attempt to construct algebraic number fields which are arithmetically equivalent but not isomorphic. Algorithms are sought to determine all solutions when p is fixed.

Suppose p and r are positive prime integers. In this note we are concerned with solutions

$$p^a = \frac{q^n - 1}{q - 1} \tag{1}$$

where a, $q = r^b$, n are positive integers, $n \geq 3$. We will write (1′) for the resulting equation when q is an arbitrary integer that is not necessarily a prime power.

Solutions of (1) have proved to be of interest in classifying finite groups having non-conjugate subgroups which induce the same permutation representation, and the associated problem of finding non-isomorphic number fields with the same Dedekind-zeta function (see for instance [**3**], [**5**], and [**9**]). When (1) has no solution for a given prime p, then any two p-complements in a finite group are conjugate ([**5**, Corollary 3.2]).

The equations

$$73 = (8^3 - 1/8 - 1) \quad \text{and} \quad 1772893 = (11^9 - 1)/(11^3 - 1)$$

show that solutions of (1) can occur when q is not itself prime. The equation $11^2 = (3^5 - 1)/(3 - 1)$ gives the only known solution of (1) when $a \geq 2$; by [**7**] it is the only solution when $a = 2$. It is proved more generally in [**7**] that the only solution of $y^2 = (x^n - 1)/(x - 1)$ in integers occur when $n = 4$, $x = 7$ or $n = 5$, $x = 3$. One of our results (Theorem 1) shows that in any solution of (1), n is prime and does not divide a. The case $a = 3$ had been settled by Nagel [**8**] and Ljunggren [**6**].

(*) Our colleague Ernst Straus passed away during the preparation of this paper, and so it could not reflect the careful attention he would have given it. We dedicate this paper to his memory. Ernst Straus had been supported in part by NSF grant MCS-82-03347.