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PROBLEMS AND RESULTS ON ADDITIVE PROPERTIES OF GENERAL SEQUENCES. I

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Dedicated to the Memory of E. G. Straus

Let $a_1 < a_2 < \cdots$ be an infinite sequence of positive integers and denote by $R(n)$ the number of solutions of $n = a_i + a_j$. The authors prove that if $F(n)$ is a monotonic increasing arithmetic function with $F(n) \to +\infty$ and $F(n) = o(n(\log n)^{-2})$ then $|R(n) - F(n)| = o((F(n))^{1/2})$ cannot hold.

1. Let $\mathscr{A} = \{a_1, a_2, \ldots\}$ $(a_1 < a_2 < \cdots)$ be an infinite sequence of positive integers. Denote by $R(n)$ the number of solutions of $n = a_i + a_j$. Sidon asked more than 50 years ago if there is a sequence $\mathscr{A}$ for which for every $n > n_0$,

$$(1) \qquad R(n) > 0 \text{ but for every } \varepsilon > 0, \qquad R(n)/n^{\varepsilon} \to 0?$$

By use of probabilistic methods P. Erdős proved the following much stronger result:

There is a sequence $\mathscr{A}$ so that there are two constants c_1 and c_2 for which for every n

$$(2) \qquad c_1 \log n < R(n) < c_2 \log n.$$

It is still a challenging problem to give a constructive proof of (2) or even of (1). We can make no contribution to this problem at the moment. An old conjecture of Erdős states that for no sequence $\mathscr{A}$ can we have

$$(3) \qquad R(n)/\log n \to c \qquad (0 < c < +\infty).$$

We cannot attack (3) at the moment but we can prove that if $G(n)$ is monotonic, $G(n) \to +\infty$ and $G(n)$ is of regular growth then

$$(4) \qquad R(n)/G(n)\log n \to c \qquad (0 < c < +\infty)$$

is possible. In view of the difficulty of (3) two questions are natural. Can one prove that for every $\mathscr{A}$

$$(5) \qquad \limsup_{n \to +\infty} |R(n) - \log n| = +\infty,$$

and is it true that for every $(0 <)\ c_1 < 1 < c_2$

$$(6) \qquad c_1 \log n < R(n) < c_2 \log n$$