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THE DARBOUX PROPERTY AND SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

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Dedicated to the memory of Ernst Straus

The most basic notion of a solution of a differential equation is that of a function that is differentiable enough to plug into it (without demanding continuity of the derivatives of highest order) and that, of course, makes the equation a true statement when you do plug it in. The lack of continuity of the derivatives has posed many obstacles in treating these solutions. In this paper we overcome several of these obstacles in the case of algebraic differential equations by using the Darboux property of derivatives.

1. Introduction. An algebraic differential equation (ADE) is one of the form:

$$P(x, y(x), y'(x), \dots, y^{(m)}(x)) = 0, \tag{1}$$

where P is a polynomial in $m + 2$ variables. We often write (1) as $P(x, \vec{y}) = 0$. In our context, x is restricted to an open or closed interval I of the real axis, and $y\colon I \to \mathbf{R}$ is a real-valued function. There is no real difficulty in extending some of our results (i.e. Theorems 1, 3, 4, and 5) to $y\colon I \to \mathbf{C}$ being a complex-valued function. An ADE in several dependent variables $y_1(x), \dots, y_n(x)$ is similarly defined to (1), but we will mention such equations only peripherally.

In discussing solutions $u(x)$ to (1) or to a system Σ of such equations, care must be taken to enunciate how smooth u is required to be—certain qualitative assertions are true for one degree of smoothness and false for another. This is the main theme of [**RUB**]. It is clear what we mean by analytic or C^∞ solutions of a system Σ of ADE's. Let us say, for convenience, that a function $u(x)$ is a *basic* solution of Σ if it (a) is differentiable enough to plug into every equation of Σ and (b) makes all these equations true statements. Following these ideas, we prove a chain of theorems with a common theme. If $u(x) \in D^m(I)$ (i.e. has derivatives of order up to and including order m on the open interval I) then about all that can be said about $u^{(m)}(x)$ is that it is a derivative. As such, it might be discontinuous a.e. [**BRU**: p. 47]. It is true that it belongs to DB_1,