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THE ANGULAR DERIVATIVE OF AN OPERATOR-VALUED ANALYTIC FUNCTION

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The classical theorem on the angular derivative of an analytic function on the half-plane $\operatorname{Re} z > 0$ is extended to operator-valued analytic functions.

1. Let Π denote the open half-plane

(1) $$\Pi = \{z \in \mathbf{C}\colon \operatorname{Re} z > 0\}.$$

For a positive number k, let Σ_k denote the set

(2) $$\Sigma_k = \{z \in \mathbf{C}\colon |\operatorname{Im} z| < k \operatorname{Re} z\}.$$

The following theorem in complex analysis is well-known:

Let f be a function analytic on Π such that $f(\Pi) \subset \Pi$. If

(3) $$a = \inf_{z \in \Pi} \frac{\operatorname{Re} f(z)}{\operatorname{Re} z},$$

then for any $k > 0$, we have

(4) $$\lim_{\substack{z \to \infty \\ z \in \Sigma_k}} \frac{f(z)}{z} = \lim_{\substack{z \to \infty \\ z \in \Sigma_k}} \frac{\operatorname{Re} f(z)}{\operatorname{Re} z} = \lim_{\substack{z \to \infty \\ z \in \Sigma_k}} f'(z) = a.$$

The limit $\lim_{z \to \infty, z \in \Sigma_k} f'(z)$ is usually called the *angular derivative* of f at ∞. The above classical theorem is the work of several mathematicians: Julia, Nevanlinna, Wolff, Carathéodory, Landau, Valiron. For the original sources, the reader is referred to [**2**, p. 216] and [**5**, p.108]. The purpose of the present paper is to extend this classical theorem to operator-valued analytic functions [**3**, pp. 92–94].

2. Throughout this paper, $\mathscr{H}$ denotes a complex Hilbert space. By an operator we always mean a bounded linear operator on $\mathscr{H}$. The identity operator is denoted by I. For an operator A on $\mathscr{H}$, the adjoint of A is denoted by A^*; the real and imaginary parts of A are denoted by $\operatorname{Re} A$ and $\operatorname{Im} A$ respectively:

$$\operatorname{Re} A = \frac{A + A^*}{2}, \quad \operatorname{Im} A = \frac{A - A^*}{2i}.$$