PACIFIC JOURNAL OF MATHEMATICS
Vol. 123, No. 1, 1986

THE ABSOLUTE GALOIS GROUP OF A PSEUDO REAL CLOSED ALGEBRAIC FIELD

DAN HARAN AND MOSHE JARDEN

The absolute Galois group of a PRC (= pseudo real closed) field is characterized as a real projective group. Specifically, it is known that if E is a PRC field, then its absolute Galois group $G(E)$ is real projective. Conversely, if G is a real projective group, then there exists a PRC field E such that $G(E) \cong G$. The construction of E makes it of infinite transcendence degree over Q. However, if a field E is algebraic over Q, then rank $G(E) \leq \aleph_0$. Therefore it is natural to ask whether for a given real projective group G of rank $\leq \aleph_0$ we may choose E to be algebraic over Q.

There are two reasons for asking this question. First of all, the corresponding question for projective groups and PAC fields is known to have an affirmative answer, since there exist algebraic PAC fields E such that $G(E) \cong \hat{F}_\omega$ = the free profinite group of ranks $\aleph_0$ and since every projective group G of rank $\leq \aleph_0$ is isomorphic to a closed subgroup of $\hat{F}_\omega$. A generalization of this fact to real projective groups and PRC fields will be a contribution to the desired description of the closed subgroups of $G(\mathbf{Q})$. Secondly, an affirmative answer to this question will give us a necessary tool to the study of the elementary theory of all PRC fields which are algebraic over Q.

The main goal of this work is indeed to give the desired affirmative answer:

THEOREM. *If K is a countable formally real Hilbertian field and G is a real projective group of* rank $\leq \aleph_0$, *then there exists a PRC algebraic extension E of K such that $G(K) \cong G$.*

In order to make this introduction self-contained we repeat the basic definitions involved in the Theorem.

A field E is said to be PRC (= pseudo real closed), if every absolutely irreducible variety V defined over K, which has a simple $\overline{K}$-rational point in every real closed field $\overline{K}$ containing K, has a K-rational point.

A diagram

$$(1)\qquad \begin{array}{ccc} & & G \\ & & \downarrow \varphi \\ B & \xrightarrow[\alpha]{} & A \end{array}$$