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ON SOLUTIONS OF DIFFERENTIAL EQUATIONS WHICH SATISFY CERTAIN ALGEBRAIC RELATIONS

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In the following, we provide another proof (Theorem 3.1 below) of recent results of Harris-Sibuya, using some elementary commutative algebra. Our purpose is to give a uniform treatment for their results which also permits some generalization. We note that the study of differential equations under the hypothesis that the solutions satisfy an algebraic relation is not new. Fano, among others, made a systematic study of this situation in the last century. Also Lamé equations in which two solutions have a rational function as their product have proved to be a good source of examples for unusual arithmetic behavior. But in the case of Harris-Sibuya, as well as the present paper, the solutions need not be solutions of the same linear equation. In the treatment below the differential equation only enters in dilineating a type of recursion.

1. Let $R = K[\{\Lambda_i\}_{i=1}^N]$ be a polynomial ring over a field K. Assume that it is graded by assigning to each variable Λ_i a natural number, $w(\Lambda_i) = w_i \in \mathbf{N}$, and then assigning

$$w\left(\prod_{i=1}^{N} \Lambda_i^{\mu_i}\right) = \sum_{i=1}^{N} \mu_i w_i.$$

The following result makes use of some elementary commutative algebra.

(1.1) THEOREM. *Let I be the ideal of R generated by a collection of polynomials, $\{f_\beta(\Lambda)\}_{\beta\in\Gamma} \subseteq R$. Let $\tilde{f}_\beta(\Lambda)$ be the leading homogeneous form of $f_\beta(\Lambda)$ with respect to the above grading; let J be the homogeneous ideal generated by $\{\tilde{f}_\beta(\Lambda)\}_{\beta\in\Gamma}$. If $Z(J) = \{(0,\ldots,0)\}$, then $Z(I)$ is a zero-dimensional variety.*

Proof. Since $Z(J) = \{(0,\ldots,0)\}$, the Nullstellansatz implies that $\sqrt{J}$, the radical of J, satisfies

$$\sqrt{J} = (\Lambda_1,\ldots,\Lambda_N).$$

Thus, for each $i \in \{1,2,\ldots,N\}$, there exists an $m_i \in \mathbf{N}$ such that $\Lambda_i^{m_i} \in J$. The sequence $\{\Lambda_1^{m_1},\ldots,\Lambda_N^{m_N}\}$ forms a regular sequence (in any order). Let