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ALTERNATIVE ALGEBRAS HAVING SCALAR INVOLUTIONS

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An involution of an algebra over a field of characteristic different from two is called *scalar* if the sum of each element and its involute is a scalar (multiple of the unit). Certain algebras having scalar involutions have played an important role in the construction of metaplectic representations and the applications of that theory to problems in number theory and automorphic forms. They also arise in an analytic context related to homomorphic discrete series and in questions about invariants of classical groups. This paper deals with determining the structure of the most general algebras having scalar involutions.

1. Non-singular subalgebras and the radical.

1.1. *Singular composition algebras.* We shall classify all algebras, including the infinite-dimensional ones, that admit a particularly restrictive type of involution. By "algebra" we mean an alternative algebra A with unit 1 over a commutative field F of characteristic different from 2. An F-linear involution $a \to a'$ of A is called *scalar* if $a + a' \in F1$ for every $a \in A$. This is equivalent to the condition: $a = a'$ precisely for $a \in F1$. It is also equivalent to the condition: $aa' \in F1$ for all $a \in A$; a is invertible if and only if $aa' \neq 0$, in which case $a^{-1} = (aa')^{-1}a'$. We normally abuse the notation to the extent of identifying $F1$ with F. With this convention, the formula

$$(1) \qquad (a \mid b) = \tfrac{1}{2}(ab' + ba')$$

$a \to a'$ being a scalar involution, defines a symmetric bilinear form an A which satisfies the law of composition

$$(2) \qquad (ab \mid ab) = (a \mid a)(b \mid b).$$

In [7], Jacobson defines a *composition algebra* as an algebra with scalar involution for which the associated form (1) is nondegenerate. We shall call such algebras *non-singular composition algebras*. Their structure has been the subject of many investigations, e.g., [1], [2], [6], [9], and is well known. They are necessarily semisimple and finite dimensional, in fact, of diffeomorphism 1, 2, 4 or 8 over the ground field. Here we drop the nondegeneracy condition and study the possibly singular case, i.e., arbitrary algebras with scalar involutions.