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EIGENVALUE BOUNDS FOR THE DIRAC OPERATOR

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A natural question in the study of geometric operators is that of how much information is needed to estimate the eigenvalues of an operator. For the square of the Dirac operator, such a question has at least peripheral physical import. When coupled to gauge fields, the lowest eigenvalue is related to chiral symmetry breaking. In the pure metric case, lower eigenvalue estimates may help to give a sharper estimate of the ADM mass of an asymptotically flat spacetime with black holes. We use three tools to estimate the eigenvalues of the square of the (purely metric) Dirac operator: the conformal covariance of the operator, a patching method and a heat kernel bound.

I. A lower bound. Let V be a vector bundle associated to the $\mathrm{SO}(n)$ ($\mathrm{Spin}(n)$) frame bundle of a compact n-dimensional oriented (spin) Riemannian manifold X, with a positive-definite inner product $\langle\ ,\ \rangle$. For each metric g, let T_g: $C^\infty(V) \to C^\infty(V)$ be a geometric elliptic symmetric differential operator of order $j < n$. If $g' = e^{2\sigma}g$ is a conformally related metric, suppose that $T_{g'} = e^{-j\sigma}e^{-(n-j)\sigma/2}T_g e^{(n-j)\sigma/2}$. Let $\lambda_1^2(g)$ denote the lowest eigenvalue of T_g^2.

PROPOSITION 1. (i) *If T_g is invertible then $\exists c > 0$ s.t. $\forall g' \in [g]$, (the conformal class of g),*

$$\lambda_1^2(g') \geq c^{-2}(\operatorname{Vol} g')^{-2j/n}. \tag{1}$$

(ii) *Suppose that a multiple mV of V contains a trivial subbundle of real dimension $> n$. Then the best constant $\tilde{c}$ in (1) is*

$$d \equiv \sup_{f\neq 0}\left|\int \left\langle f, T_g^{-1}f\right\rangle d\operatorname{vol}\right| / \| f \|^2_{2n/(n+j)}.$$

Proof. (i) Let ψ range through $C^\infty(V)$. Then

$$\begin{aligned}\lambda_1^{-1}(g') &= \sup_{\psi\neq 0}\left|\int \left\langle \psi, T_{g'}^{-1}\psi\right\rangle d\operatorname{vol}'\right| \Big/ \int \langle \psi, \psi\rangle\, d\operatorname{vol}' \\ &= \sup_{\psi\neq 0}\left|\int e^{n\sigma}\left\langle \psi, e^{-(n-j)\sigma/2}T_g^{-1}e^{(n+j)\sigma/2}\psi\right\rangle d\operatorname{vol}\right| \Big/ \int e^{n\sigma}\langle \psi, \psi\rangle\, d\operatorname{vol} \\ &= \sup_{f\neq 0}\left|\int \left\langle f, T_g^{-1}f\right\rangle d\operatorname{vol}\right| \Big/ \int e^{-j\sigma}\langle f, f\rangle\, d\operatorname{vol}.\end{aligned}$$