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LIFTING UNITS IN SELF-INJECTIVE RINGS AND AN INDEX THEORY FOR RICKART C^*-ALGEBRAS

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In this paper we study the following question: ***If R is a right self-injective ring and I an ideal of R, when can the units of R/I be lifted to units of R?***

We answer this question in terms of $K_0(I)$. For a purely infinite regular right self-injective ring R we obtain an isomorphism between $K_1(R/I)$ and $K_0(I)$ which can be viewed as an analogue of the index map for Fredholm operators.

By giving a purely algebraic description of the connecting map $K_1(A/I) \to K_0(I)$ in the case where A is a Rickart C^*-algebra, we are able to extend the classical index theory to Rickart C^*-algebras in a way which also includes Breuer's theory for W^*-algebras.

0. Preliminary results. Throughout this paper R will denote an associative ring with 1. By a *rng* we mean a ring which does not necessarily have a 1.

We write $M_n(R)$ for the ring of all $n \times n$ matrices over R, and $\mathrm{GL}_n(R)$ for the group of units of $M_n(R)$, though we shall write $U(R)$ rather than $\mathrm{GL}_1(R)$. For $1 \leq i,\ j \leq n$ let $e_{ij} \in M_n(R)$ be the usual matrix units. Define $E_n(R)$ to be the subgroup of $\mathrm{GL}_n(R)$ generated by all the matrices of the form $1 + re_{ij}$, $r \in R$, $i \neq j$; and $GE_n(R)$ to be the subgroup of $\mathrm{GL}_n(R)$ generated by $E_n(R)$ together with the subgroup $D_n(R)$ of all invertible diagonal matrices. If $GE_n(R) = \mathrm{GL}_n(R)$, then we say that R is a *GE_n-ring*; if R is a GE_n-ring for all $n > 1$ then R is said to be a *GE-ring*.

If R is a GE_n-ring, then $E_n(R)$ is a normal subgroup of $\mathrm{GL}_n(R)$ and hence $\mathrm{GL}_n(R) = D_n(R)E_n(R)$.

Let $\mathrm{GL}(R)$ denote the direct limit of the directed system

$$U(R) \to \mathrm{GL}_2(R) \to \mathrm{GL}_3(R) \to \cdots$$

where each $a \in \mathrm{GL}_n(R)$ is mapped to

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$