PACIFIC JOURNAL OF MATHEMATICS
Vol. 128, No. 1, 1987

THE PICARD NUMBERS OF ELLIPTIC SURFACES WITH MANY SYMMETRIES

PETER F. STILLER

In this paper we compute the Picard numbers of several families of elliptic surfaces (see Example 1, §5 for a typical result.) This is equivalent to the difficult problem of determining the rank of the Mordell-Weil group of certain elliptic curves over function fields. Our method is to study the action induced by automorphisms of these surfaces on a relevant part of the cohomology. The cohomology classes are represented by certain inhomogeneous differential equations—our so-called inhomogeneous de Rham cohomology—where the effect of the action is easily understood.

1. An overview. A complex surface is said to be elliptic if it can be mapped onto a curve in such a way that the general fiber is a curve of genus one (see Kodaira [**9**] and [**10**]). In this paper we focus on computing the Picard number of certain surfaces of this type. Recall that the Picard number is defined to be the rank of the Néron-Severi group of the surface—that is, the group of divisors modulo algebraic equivalence—which is known to be a finitely generated abelian group.

Let E be an elliptic surface and denote by $\pi\colon E \to X$ a projection of E onto a curve X with generic fiber E^{gen} a curve of genus one over the function field $K(X)$ of X. We shall assume that $\pi\colon E \to X$ has a section $\wp\colon X \to E$, $\pi \circ \wp = 1_X$, that the J-invariants of the fibers are not constant, and that there are no exceptional curves of the first kind in the fibers. Let $S \subset X$ be the finite set of points at which the family E/X degenerates—that is, where $\pi^{-1}(s)$ fails to be an elliptic curve. (Note that there are no multiple fibers.) The degenerate fiber types are classified (see Kodaira [**9**]) and we shall label the types following Kodaira. We denote by $\mathrm{NS}(E)$ the Néron-Severi group of E and by ρ_E its rank which is called the Picard number of E.

The group $\mathrm{NS}(E)$ is naturally a subgroup of $H^2(E,\mathbf{Z})$—both are torsion free in our case (see Cox and Zucker [**1**])—and includes in $H^1(E,\Omega^1_E)$ the $(1,1)$ part of the Hodge decomposition of the cohomology,