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CHARACTERIZING REDUCED WITT RINGS OF HIGHER LEVEL

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Mulcahy's Spaces of Signatures (SOS) is an abstract setting for the reduced Witt rings of higher level of Becker and Rosenberg just as Marshall's Spaces of Orderings is an abstract setting for the ordinary reduced Witt ring. Finitely constructible SOS's are those built up in a finite number of steps from the smallest SOS using 2 operations. We show that finitely constructible SOS's are precisely those that arise from preordered fields (subject to a certain finiteness condition). This allows us to give an inductive construction for the reduced Witt rings of higher level for certain preordered fields, which generalizes a result of Craven for the ordinary reduced Witt ring. We also obtain a generalization of Bröcker's results on the possible number of orderings of a field.

1. Preliminaries. For a field K we set $\dot{K} = K \setminus \{0\}$. The symbol $\sqcup$ stands for disjoint union. We begin by recalling some of the theory of preorders of higher level from [**4**]:

A subset T of K is a preorder if $\dot{T} = T \setminus \{0\}$ is a subgroup of $\dot{K}$ and $\dot{T} + \dot{T} \subseteq \dot{T}$. We assume throughout that all preorders are of finite exponent, i.e., $K^m \subseteq T$ for some $m \in \mathbf{N}$. Since $-1 \notin \dot{T}$, the exponent of the group $\dot{K}/\dot{T}$ is even, say $2n$. We call n the level of T.

Let $\mu = \{z \in \mathbf{C}\colon z^r = 1 \text{ for some } r \in \mathbf{N}\}$. For an abelian group G let G^* denote $\mathrm{Hom}(G, \mu)$, with the usual compact-open topology. $\chi \in (\dot{K})^*$ is called a signature if $\ker\chi + \ker\chi \subseteq \ker\chi$. We write $\mathrm{Sgn}(K)$ for the set of signatures of K. For a preorder T let $X_T = \{\chi \in \mathrm{Sgn}(K)\colon \chi(\dot{T}) = 1\}$. Then $\dot{T} = \bigcap_{\chi \in X_T} \ker\chi$ [**4**, 1.4].

We make extensive use of Krull valuations: If $v\colon \dot{K} \to \Gamma$ is a valuation, we denote the valuation ring by A, the group of units by U, the maximal ideal by I and the residue class field by k. If k is formally real, we say v is a real valuation.

An element $\chi \in (\dot{K})^*$ is "compatible" with a valuation v, written $v \sim \chi$, if $1 + I \subseteq \ker\chi$. In this case, the equation $\bar{\chi}(u + I) = \chi(u)$ defines an element $\bar{\chi} \in (\dot{k})^*$, called the pushdown of χ along v, and $\chi \in \mathrm{Sgn}(K)$ iff $\bar{\chi} \in \mathrm{Sgn}(k)$ [**3**, 1.12, 2.5]. A preorder T of K is "fully compatible", written $v \sim_f T$, if each $\chi \in X_T$ is compatible with v, i.e., if $1 + I \subseteq T$. In that case the image of $A \cap T$ in k, denoted $\bar{T}$, is a preorder of k.