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SOME EXPLICIT UPPER BOUNDS ON THE CLASS NUMBER AND REGULATOR OF A CUBIC FIELD WITH NEGATIVE DISCRIMINANT

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Explicit upper bounds are developed for the class number and the regulator of any cubic field with a negative discriminant. Lower bounds on the class number are also developed for certain special pure cubic fields.

1. Introduction. Let $\mathscr{K}$ be any cubic number field with discriminant $\Delta < 0$ and regulator R. Since either $4 \mid \Delta$ or $\Delta \equiv 1 \pmod 4$, we may assume that $\Delta = df^2$, where d is the discriminant of a quadratic field. Further, since $d < 0$ and either $4 \mid d$ or $d \equiv 1 \pmod 4$, we must have $|d| \geq 3$. Let $\mathcal{O}_{\mathscr{K}}$ be the ring of all algebraic integers of $\mathscr{K}$ and let h be the number of ideal classes of $\mathcal{O}_{\mathscr{K}}$.

From a classical, general result of Landau [**11**] we know that

$$hR = O\left(\sqrt{|\Delta|}\,(\log|\Delta|)^2\right).$$

More recently Siegel [**19**] and Lavrik [**13**] have given general results from which an explicit constant c can be easily determined such that

$$hR < c\sqrt{|\Delta|}\,(\log|\Delta|)^2.$$

However, in the case of a pure cubic field ($d = -3$), Cohn [**6**] has shown that

$$hR = O\left(\sqrt{|\Delta|}\log|\Delta|\log\log|\Delta|\right).$$

In this paper we will develop an explicit upper bound on hR which depends on d and f ($= \sqrt{\Delta/d}$). In the pure cubic case our results give

$$hR < \frac{\sqrt{|\Delta|}}{6\sqrt{3}}\log|\Delta|.$$

We make use of the well-known fact that

$$\Phi(1) = \lim_{s\to 1}\frac{\zeta_{\mathscr{K}}(s)}{\zeta(s)} = h\kappa,$$

where

$$\kappa = CR \quad\text{and}\quad C = 2\pi/\sqrt{|\Delta|}\,.$$