

ANALYSIS OF INVARIANT MEASURES IN DYNAMICAL SYSTEMS BY HAUSDORFF MEASURE

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Hausdorff measure is a preliminary concept in the definition of Hausdorff dimension, which is one concept of the degree of singularity of a finite measure. In general, Hausdorff measure does not permit as detailed an analysis of an arbitrary natural invariant measure arising from a dynamical system as Lebesgue measure permits of an absolutely continuous measure. It is shown that even for a dynamical system as simple as a modified baker's transformation, the natural invariant measure has no representation as an indefinite integral with respect to any Hausdorff measure. However, Hausdorff measure can be used to compare different natural invariant measures according to degree of singularity even when their Hausdorff dimensions are identical.

1. Introduction. In this article we seek to illustrate some of the capabilities and limitations of Hausdorff measure for the analysis of invariant measures in dynamical systems in more detail than is possible with Hausdorff dimension alone.

Hausdorff dimension is a concept of the size of a set or the degree of singularity of a measure. In recent years it has often been used for the study of dynamical systems because many of the sets of interest which arise are of Lebesgue measure zero and many of the measures of interest which arise are singular with respect to Lebesgue measure. Thus Lebesgue measure is of little or no aid in the analysis of these sets and measures. The Hausdorff dimension of a set of Lebesgue measure zero or a measure singular with respect to Lebesgue measure can, however, range over a wide spectrum of possible values; such sets and measures can thus be distinguished and classified by their Hausdorff dimensions.

Hausdorff measure is a necessary preliminary to the definition of Hausdorff dimension and in cases where the Hausdorff dimension fails to distinguish between two sets or between two measures (their Hausdorff dimensions being the same) it is sometimes possible to compare their sizes using Hausdorff measure.