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THREE QUAVERS ON UNITARY ELEMENTS IN C^*-ALGEBRAS

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Henry Dye in memoriam

Unitary polar decomposition of elements in C^*-algebras is discussed in relation to the theory of unitary rank; and characterizations of algebras admitting weak or unitary polar decomposition of every element are given.

Introduction. Let A be a unital C^*-algebra, and denote by $\mathrm{GL}(A)$ and $\mathscr{U}(A)$ the groups of invertible and unitary elements in A, respectively. The set

$$\mathscr{P}(A) = \mathscr{U}(A)A_+$$

consists of those elements that admit a unitary polar decomposition in A. The formulae $x = (x|x|^{-1})|x|$ and $x = u|x| = \lim u(|x|+n^{-1})$ show that $\mathrm{GL}(A) \subseteq \mathscr{P}(A)$ and that $\mathrm{GL}(A)$ is dense in $\mathscr{P}(A)$. Moreover, it was shown in **[12]** and **[16]** that each element in A has a canonical approximant in $\mathscr{P}(A)^=$.

We know from Mazur's theorem that $\mathrm{GL}(A) = A\backslash\{0\}$ only if $A = \mathbf{C}$. The corresponding question, when $\mathscr{P}(A) = A$, is more subtle, and will be addressed in the third of these short notes. In the first two we shall study certain phenomena in the unit ball A^1 of A. In particular we shall be concerned with the set

$$\mathscr{P}(A)^1 = \mathscr{U}(A)A_+^1.$$

(As usual we write S^1 for $S \cap A^1$, for any subset S of A.) It is quite easy to see that

$$\mathrm{GL}(A)^1 \subseteq \mathscr{P}(A)^1 \subseteq \tfrac{1}{2}(\mathscr{U}(A) + \mathscr{U}(A)),$$

and that these sets are dense in one another. By **[16**, Proposition 3.16] their common closure $(\mathscr{P}(A)^1)^=$ consists of those elements x in A such that for every $\varepsilon > 0$ there are unitary elements u_1, u_2 and u_3 with $x = \frac{1}{2}(1-\varepsilon)u_1 + \frac{1}{2}(1-\varepsilon)u_2 + \varepsilon u_3$.

1. Unitary rank revisited. Based on the Russo-Dye theorem **[17]**, the theory of unitary rank is the discussion of the least number of unitaries