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MIXING AUTOMORPHISMS OF COMPACT GROUPS AND A THEOREM BY KURT MAHLER

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Dedicated to the memory of Henry Abel Dye

We investigate the higher order mixing properties of $\mathbb{Z}^d$-actions by automorphisms of a compact, abelian group and exhibit a connection between certain mixing conditions and a result by Kurt Mahler.

1. Introduction. Let X be a compact, abelian group, and let $\mathrm{Aut}(X)$ denote the group of continuous automorphisms of X. We investigate the mixing behaviour of $\mathbb{Z}^d$-actions $\alpha\colon \mathbf{n} \to \alpha_{\mathbf{n}}$ on X with the property that $\alpha_{\mathbf{n}} \in \mathrm{Aut}(X)$ for every $\mathbf{n} \in \mathbb{Z}^d$ (such an action will be called a $\mathbb{Z}^d$-action by automorphisms). If (X, α) satisfies the *descending chain condition*, i.e. if every decreasing sequence of closed, α-invariant subgroups of X eventually becomes constant, then α is algebraically and topologically conjugate to the shift action on a closed, shift invariant subgroup of $(\mathbb{T}^k)^{\mathbb{Z}^d}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and is automatically a *Markov shift* in d dimensions (cf. **[KS]** for a more general result). Furthermore it is easy to see that the dual group $\hat{X}$ of X can naturally be viewed as a finitely generated R_d-module, where R_d is the ring of Laurent polynomials in d variables with integral coefficients (cf. **[KS]**). In view of this correspondence between finitely generated R_d-modules and $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups the question arises how the algebraic properties of the R_d-module $M = \hat{X}$ reflect the dynamical properties of the $\mathbb{Z}^d$-action α. In **[S2]** it was shown how to read off ergodicity, mixing, expansiveness, and certain facts about periodic orbits, from properties of the prime ideals associated with the R_d-module M. In this paper we continue this investigation and study the higher order mixing behaviour of such actions. This problem was raised by a paper of F. Ledrappier which contains examples of such actions which are (strongly) mixing, but which fail to be r-mixing for some $r \geq 2$. In these examples higher order mixing breaks down in a particularly interesting way: there exist a nonempty set $S \subset \mathbb{Z}^d$ and Borel sets $\{B_{\mathbf{n}} \subset X : \mathbf{n} \in S\}$ with positive Haar measure such that the sets $\{\alpha_{k\mathbf{n}}(B_{\mathbf{n}}) : \mathbf{n} \in S\}$ fail to become asymptotically independent as $k \to \infty$. In order to simplify terminology we call the set S a *mixing*