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DISTANCE BETWEEN UNITARY ORBITS IN VON NEUMANN ALGEBRAS

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Dedicated to Professor Shozo Koshi on his 60th birthday

Let $\mathscr{M}$ be a semifinite factor. For normal operators x and y in $\mathscr{M}$, introducing the spectral distance $\delta(x,y)$, we show that $\delta(x,y) \geq \operatorname{dist}(\mathscr{U}(x),\mathscr{U}(y)) \geq c^{-1}\delta(x,y)$ with a universal constant c, where $\operatorname{dist}(\mathscr{U}(x),\mathscr{U}(y))$ denotes the distance between the unitary orbits $\mathscr{U}(x)$ and $\mathscr{U}(y)$. The equality $\operatorname{dist}(\mathscr{U}(x),\mathscr{U}(y)) = \delta(x,y)$ holds in several cases. Submajorizations are established concerning the spectral scales of τ-measurable selfadjoint operators affiliated with $\mathscr{M}$. Using these submajorizations, we obtain the formulas of L^p-distance and anti-L^p-distance between unitary orbits of τ-measurable selfadjoint operators in terms of their spectral scales. Furthermore the formulas of those distances in Haagerup L^p-spaces are obtained when $\mathscr{M}$ is a type III_1 factor. The appendix by H. Kosaki is the generalized Powers-Størmer inequality in Haagerup L^p-spaces.

Introduction. It is an interesting problem in matrix theory to estimate distances between unitary orbits of matrices by their eigenvalues. Let A and B be $n \times n$ normal matrices whose eigenvalues are $\alpha_1,\dots,\alpha_n$ and $\beta_1,\dots,\beta_n$, respectively, with multiplicities counted. Let $\operatorname{dist}(\mathscr{U}(A),\mathscr{U}(B))$ denote the distance between the unitary orbits $\mathscr{U}(A)$ and $\mathscr{U}(B)$. The optimal matching distance between the eigenvalues of A and B is given by

$$\delta(A,B) = \min_{\pi}\max_{i} |\alpha_i - \beta_{\pi(i)}|,$$

where π runs over all permutations of $\{1,\dots,n\}$. Then

$$\operatorname{dist}(\mathscr{U}(A),\mathscr{U}(B)) \leq \delta(A,B)$$

is immediate. Bhatia, Davis and McIntosh **[9]** proved that

$$\operatorname{dist}(\mathscr{U}(A),\mathscr{U}(B)) \geq c^{-1}\delta(A,B)$$

with a universal constant c. A difficult and still open conjecture is that $\operatorname{dist}(\mathscr{U}(A),\mathscr{U}(B)) = \delta(A,B)$ holds for every pair of normal matrices A and B (i.e. $c = 1$). But this equality was proved to hold for several classes of normal matrices (see **[7, 10, 21, 41, 45]**). The analogous