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HORROCKS' QUESTION FOR MONOMIALLY GRADED MODULES

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In this paper several related inequalities are developed which, when appropriately specialized, yield affirmative answers to Horrocks' question in the case of multi-graded $R[X_1, \ldots, X_n]$-modules for certain rings R.

1. Introduction. In 1978 Hartshorne reported a question [**H**, Problem 24] due to Horrocks which essentially asks whether the ith Betti number of a finite length module over an n-dimensional regular local ring is at least as large as the corresponding binomial coefficient $\binom{n}{i}$. Recall that the ith Betti number of a module M is defined to be the rank of an ith syzygy of M with respect to a minimal free resolution. Equivalently, if (R, m, k) is a regular local ring, the question asks whether $\dim_k \operatorname{Tor}_i^R(k, M) \geq \binom{n}{i}$. Evans and Griffith [**EG$_2$**] give an affirmative answer to the conjecture for finite length modules over the polynomial ring $k[X_1, \ldots, X_d]$, with k a field, which are direct sums of cyclic modules $k[X_1, \ldots, X_d]/I$, where I is generated by monomials.

One of the principal results of this paper is contained in Theorem 3.3 which gives a quite general inequality relating values of an additive function defined on a class $\mathscr{C}$ of $R[X_1, \ldots, X_d]$-modules satisfying some very reasonable closure conditions.

Seemingly more stringent conditions must be put on the modules themselves, namely admission of a "high-low decomposition", but at the moment it is unclear just how restrictive this condition is. However the base ring R need only be commutative with identity.

An easy application of Theorem 3.3 extends the result of Evans and Griffith to the larger class of all finite length modules graded by monomials over some regular local (or graded) based ring R for which Horrocks' inequality is known to hold, e.g. $\mathbb{Z}^d$-graded over $R[X_1, \ldots, X_d]$ with R regular and Krull $\dim R \leq 4$.

In §4 the techniques used in the proof of Theorem 3.3 are employed in a somewhat different fashion to obtain lower bounds for images of certain maps of Koszul homology. In this instance it will be seen that a high-low decomposition is used in a way quite different than in