PACIFIC JOURNAL OF MATHEMATICS
Vol. 142, No. 2, 1990

EXTENSION THEOREMS FOR FUNCTIONS OF VANISHING MEAN OSCILLATION

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A locally integrable function is said to be of vanishing mean oscillation (VMO) if its mean oscillation over cubes in $\mathbf{R}^d$ converges to zero with the volume of the cubes. We establish necessary and sufficient conditions for a locally integrable function defined on a bounded measurable set of positive measure to be the restriction to that set of a VMO function.

1. Introduction. Let F be a locally integrable function on $\mathbf{R}^d$ and let Q be a cube in $\mathbf{R}^d$ with sides parallel to the axes. (We denote the set of all such cubes in $\mathbf{R}^d$ by $\mathfrak{F}'$.) We denote the Lebesgue measure of Q by $|Q|$ and the length of Q by $l(Q)$. We denote the average of F on Q by F_Q; that is $F_Q = \frac{1}{|Q|}\int_Q F\,dt$. We say F is of bounded mean oscillation (abbreviated BMO($\mathbf{R}^d$) or simply BMO) if

$$\sup_{Q\in\mathfrak{F}'} \frac{1}{|Q|}\int_Q |F - F_Q| < \infty. \tag{1.1}$$

We denote this supremum by $\|F\|_*$. $\|\ \|_*$ defines a norm on BMO and BMO is a Banach space with respect to this norm. (We identify functions which differ by a constant.) If in (1.1) we restrict the cubes to be dyadic we obtain the space dyadic-BMO and we denote the corresponding norm by $\|\ \|_{*,d}$. (By a dyadic cube we mean a cube of the form $Q = \{k_j < x_j < (k_j + 1)2^{-n}; 1 \le j \le d\}$ where n and k_j, $1 \le j \le d$, are integers.) We will denote the set of dyadic cubes of length 2^{-n} by D_n and Q_0 will denote the dyadic unit cube. The function space BMO was introduced in 1961 by John and Nirenberg [7] who proved the following fundamental theorem:

THEOREM 1.1. *Let F be a locally integrable function on $\mathbf{R}^d$, and for each $n \in \mathbf{Z}$ define*:

$$\overline{\mu}_n(F) = \inf\left\{\frac{1}{\lambda}:\ \sup_{l(Q)\le 2^{-n}}\ \inf_{a\in\mathbf{R}} \frac{1}{|Q|}\int_Q e^{\lambda|F-a|} < 2\right\}.$$

Then,

(1) $F \in$ BMO *if and only if*

(2) $\sup_{n\in\mathbf{Z}} \overline{\mu}_n(F) < \infty$.