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A DISCRETE LYAPUNOV FUNCTION FOR A CLASS OF LINEAR DIFFERENTIAL EQUATIONS

HAL L. SMITH

An integer-valued function on $\mathfrak{R}^n$ is shown to decrease along trajectories of certain linear systems of ordinary differential equations.

1. Introduction. Recently, there has been a greal deal of success in obtaining very striking results on the global behavior of both finite and infinite dimensional dynamical systems through the use of integer-valued Lyapunov functions which decrease in value along trajectories. Matano [6] uses the lap number (originally discovered by Nickel [7]), a measure of the number of zeros of a solution of a scalar reaction diffusion equation on an interval, and the fact that it does not increase, to show that solutions converge to equilibria. Henry [3] and Angenent [1] exploit this further to obtain the Morse-Smale property for scalar reaction diffusion equations. For functional differential equations, Mallet-Paret [4] shows that a count of the number of zeros of a solution in an interval is an integer valued Lyapunov function for a certain class of equations. He exploits this fact to obtain a Morse decomposition of the global attractor. Smillie [8] uses the fact that the number of sign changes in the components $\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n$ does not increase with time for cooperative, tridiagonal systems of ordinary differential equations to conclude that all bounded solutions converge. Fusco and Oliva [2] use a similar integer-valued Lyapunov function to identify a class of ordinary differential equations with the Morse-Smale property. The author and Mallet-Paret [5] use such a function to obtain a Poincaré-Bendixson type result for a class of ordinary differential equations in $\mathfrak{R}^n$, $n \geq 2$. It is clear that these various (and related) integer-valued Lyapunov functions put severe constraints on the global asymptotic behavior of the various systems.

We note that for many of the applications cited above, the crucial point is to have an appropriate integer valued Lyapunov function for the variational equation along an orbit, or for a class of linear systems which contain the variational equation. For then one can argue that the Lyapunov function decreases along the difference of two solutions.