PACIFIC JOURNAL OF MATHEMATICS
Vol. 147, No. 1, 1991

# $D$-HARMONIC DISTRIBUTIONS AND GLOBAL HYPOELLIPTICITY ON NILMANIFOLDS

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**Let $M = \Gamma\backslash N$ be a compact nilmanifold. A system of differential operators $D_1, \dots, D_k$ on $M$ is *globally hypoelliptic* (GH) if when $D_1 f = g_1, \dots, D_k f = g_k$ with $f \in \mathscr{D}'(M)$, $g_1, \dots, g_k \in C^\infty(M)$ then $f \in C^\infty(M)$. Let $X_1, \dots, X_k$ be real vector fields on $M$ induced by the Lie algebra $\mathscr{N}$ of $N$. We study the relationships between (GH) of the system $X_1, \dots, X_k$ on $M$, (GH) of the operator $D = X_1^2 + \cdots + X_k^2$, the constancy of $D$-harmonic distributions on $M$, and related algebraic conditions on $X_1, \dots, X_k \in \mathscr{N}$.**

**0. Introduction.** Let $M = \Gamma\backslash N$ be a compact nilmanifold, where $N$ is a connected, simply connected real nilpotent Lie group with a discrete subgroup $\Gamma$. There is a unique probability measure $\mu$ defined on the Borel sets on $M$ and invariant under the action of $N$ on $M$ by right translations. Every $\mu$-integrable function $f$ on $M$ defines a distribution by the formula $(f, \phi) = \int_M f\phi\, d\mu$, $\phi \in C^\infty(M)$. Let $\mathscr{N}$ be the Lie algebra of $N$. If $X \in \mathscr{N}$ then $X$ induces a vector field (which we will denote also by $X$) on $\Gamma\backslash N$ by $(Xf)(\Gamma n) = (d/dt)|_{t=0} f(\Gamma n \exp tX)$. Consider the left-invariant sum of squares of such vector fields $X_1, \dots, X_k \in \mathscr{N}$. This second order differential operator $D = X_1^2 + \cdots + X_k^2$ can be regarded as acting on the right on distributions on $\Gamma\backslash N$. A distribution $u \in \mathscr{D}'(M)$ is *D-harmonic* if $Du = 0$ on $M$. The operator $D$ is *globally hypoelliptic* (GH) if when $Df = g$ with $f \in \mathscr{D}'(M)$, $g \in C^\infty(M)$, then $f \in C^\infty(M)$. The *system* of vector fields $X_1, \dots, X_k$ on $M$ is (GH) if when $X_1 f = g_1, \dots, X_k f = g_k$ with $f \in \mathscr{D}'(M)$, $g_1, \dots, g_k \in C^\infty(M)$, then $f \in C^\infty(M)$. In this paper we investigate relationships between (GH) of $D$, (GH) of the corresponding system $X_1, \dots, X_k$ of vector fields, the constancy of $D$-harmonic distributions on $M$, and related algebraic conditions on $X_1, \dots, X_k \in \mathscr{N}$.

Our results are summarized in the figure below. In this figure, functionals $\Lambda \in \mathscr{N}_j^*$ are assumed to be *integral*, i.e. $\Lambda(\log\Gamma \cap \mathscr{N}_j) \subseteq \mathbb{Z}$; $\mathscr{N} = \mathscr{N}_1 \supset \mathscr{N}_2 \supset \cdots \supset \mathscr{N}_r \supset \mathscr{N}_{r+1} = \{0\}$ is the *lower central series* of $\mathscr{N}$ (we say $\mathscr{N}$ is of *step* $r$), and $\mathscr{L}$ is the subalgebra of $\mathscr{N}$ Lie-generated by $X_1, \dots, X_k$. Let $\mathscr{W}_\pi$ be an ideal in $\ker(d\pi)$ such that