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SOME REMARKS ON ORDERINGS UNDER FINITE FIELD EXTENSIONS

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Let X_K denote the space of orderings of a field K, and $r_{L/K}\colon X_L \to X_K$ the restriction mapping, when L/K is a field extension. Fixing K, the image sets $r_{L/K}(X_L)$ for finite extensions L/K are investigated. If K is hilbertian, any clopen subset $U \subset X_K$ has the form $U = r_{L/K}(X_L)$ for some finite L/K, and $[L : K]$ can be bounded in terms of U. This bound is even sharp in some cases, but not always. A second construction gives the same qualitative result for a much larger class of fields. It is based on iterated quadratic extensions. The bounds on $[L : K]$ obtained here are weaker than in the hilbertian case.

Let K be a field, and let X_K be the (topological) space of its orderings. It is known to be compact and totally disconnected. If L/K is a finitely generated field extension, then Elman, Lam and Wadsworth showed that the natural restriction mapping $r = r_{L/K}\colon X_L \to X_K$ is (not only closed but also) open [**ELW**, Theorem 4.9]. In particular, the set $r_{L/K}(X_L)$ of those orderings of K which extend to L is clopen (:= closed and open) in X_K. This means that it is a union of finitely many *basic* clopen subsets, i.e. sets of the form

$$X_K(a_1, \dots, a_t) := \{x \in X_K : a_1, \dots, a_t \text{ are non-negative in } x\}$$

with $a_i \in K$. Conversely, given a clopen subset U of X_K, it is not hard to find explicitly a finitely generated extension L/K such that $U = r_{L/K}(X_L)$. For example, if the complement of U is presented as

$$X_K \backslash U = \bigcup_{\iota=1}^{s} X_K(a_1^{\iota}, \dots, a_{t_\iota}^{\iota}),$$

then one may take $L = K(\phi_1, \dots, \phi_s)$ where ϕ_ι is the Pfister form $\langle 1, a_1^{\iota}\rangle \otimes \cdots \otimes \langle 1, a_{t_\iota}^{\iota}\rangle$ [**ELW**, Theorem 4.18].

The question becomes somewhat harder when one tries to realize U by a *finite* extension L/K. In fact, this is not always possible [**ELW**, §5]. On the other hand, Prestel has shown [**Pr**, p. 904] that it is possible if the field K is hilbertian. In fact, this is merely a