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THE BOUNDARY OF A SIMPLY CONNECTED DOMAIN AT AN INNER TANGENT POINT

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Let T^* be the set of accessible boundary points at which the inner tangent to ∂D exists. That is, if $a \in T^*$ and $w(a)$ represents its complex coordinate, then there exists a unique $\nu(a)$, $0 \le \nu(a) < 2\pi$, such that for each $\varepsilon > 0$ $(\varepsilon < \frac{\pi}{2})$ there exists a $\delta > 0$ such that

$$\Delta = \left\{w(a) + \rho e^{i\nu} : 0 < \rho < \delta,\, |\nu - \nu(a)| < \frac{\pi}{2} - \varepsilon\right\} \subset D$$
$$\text{and } w \to a \text{ as } w \to w(a), \qquad w \in \Delta.$$

Let $\gamma(a, r)$ represent the unique component of $D \cap \{|w - w(a)| = r\}$ that intersects the inner normal $\{w(a) + \rho e^{i\nu(a)} : \rho > 0\}$, $L(a, r)$ denote the length of $\gamma(a, r)$ and set $A(a, r) = \int_0^r L(a, r')\, dr'$. Finally let AD^* be those points of T^* at which a non-zero angular derivative exists.

Our main result is a *purely geometrical* proof of a theorem that describes the boundary of D near $a \in T^*$. As a consequence we have

(1) a geometric description of the boundary of D near almost every $a \in AD^*$ that is a generalization of the geometric behavior of a smooth curve,

(2) an answer on T^* and hence on AD^* of the two open questions and conjectures made by McMillan in [3, p. 739] concerning the length and area ratios

$$\frac{L(a, r)}{2\pi r} \quad \text{and} \quad \frac{A(a, r)}{\pi r^2} \quad \text{as } r \to 0.$$

1. Introduction.

1.1. Many of the definitions introduced in §§1.1 to 1.3 can be found in McMillan's papers.

Let D be a simply connected plane domain, not the whole plane, and define on D the relative metric d_D, the relative distance between two points of D being defined as the infimum of the Euclidean diameters of curves that lie in D and join these two points. Let (D^*, d_{D^*}) be the completion of the metric space (D, d_D). Now $D^* = D \cup A^*$ where D is an isometric copy of D in D^* and A^* is the set of accessible boundary points of D. Any limits involving accessible boundary points are taken in d_{D^*}.